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BOOK REVIEW

Global classical solutions for nonlinear evolution equations, by Li Ta-Tsien and Chen Yunmei. Longman Scientific and Technical, Harlow, 1992, xii+209 pp., \$69.00. ISBN 0-58205588-1

Most dynamical problems in physics are governed by evolution equations. Evolution equations are partial differential equations whose solutions one studies as functions of a distinguished independent variable t, called *time*. Examples are heat flow or diffusion governed by parabolic equations, vibration or electromagnetism governed by hyperbolic equations, and quantum mechanics governed by Schrödinger equations. These are the three types of equations studied in this book. Of course, there are many other famous evolution equations such as the Navier-Stokes equations of incompressible fluids, the Yang-Mills equations in Minkowski space, reaction-diffusion equations, hyperbolic conservation laws, the Korteweg-de Vries equation, and so on.

The basic problem is to understand what happens when "arbitrary" initial data are specified, say at $t = 0$. The mathematician's job is to specify the word "arbitrary" and the properties of the solutions. Specifically, one asks for (1) local existence (to find classes of functions for which at least one solution exists for at least a nontrivial interval of time), (2) global existence (to find classes for which solutions exist for all time $0 \leq t < \infty$), (3) uniqueness (to find function classes in which there is at most one solution with given initial data), (4) regularity of the solutions (differentiability, etc.), (5) singularities (Which initial data lead to singular solutions, and where are the singularities located?), (6) asymptotics (What is the behavior as $t \to \infty$?), (7) scattering (the relation between behavior as $t \to -\infty$ and $t \to +\infty$), (8) periodic solutions, traveling waves, and other special solutions, (9) stability properties under various perturbations for finite or infinite times, and (10) bifurcation under changes of parameters.

Nonlinear partial differential equations are notoriously difficult creatures. Up to fairly recent times the few successes included the Cauchy-Kovalevsky Theorem, Riemann's early study of shock waves, and Leray's analysis of the Navier-Stokes equations in the 1930s. Since 1950 the influence of functional analysis has been profound. This influence was in most cases more psychological than substantive. In particular, the notion of a distribution allowed us to clarify the concept of what a generalized (not necessarily differentiable) solution is. In some cases a concept as simple as the contraction principle (that in a metric space every contraction has a fixed point) could be employed in unexpectedly subtle ways. Functional analysis in general helped us unify the multifarious equations and methods in partial differential equations. Some modern techniques which have been developed in the last thirty years include Nash-Moser techniques, monotone and accretive operator theory, nonlinear semigroup theory, and the compensated and concentrated compactness methods.

One area of rapid development has been the study of the *small* (i.e., small amplitude) solutions. (This is sometimes called the weakly nonlinear theory.) If a partial differential equation contains the nonlinear term $f(u)$, where $u = u(t, x, \dots)$ denotes a solution, and if the values under consideration satisfy $|u(t, x, \dots)| < \varepsilon$, then obviously the only behavior of f that matters is for small u. Therefore, assuming f is a smooth function, only the leading term in the Taylor expansion of f matters. If we write the expansion as $f(u) = cu^p + O(u^{p+1})$, then only the degree p matters. This book is concerned with the question of the global existence of *regular* (i.e., highly differentiable) *small* solutions of nonlinear heat and wave equations. That is, the initial data are small and regular but are otherwise arbitrary, and they lead to the construction of regular, small, global solutions. Only the degrees of the nonlinear terms at $u = 0$ are relevant, and almost no additional structure assumptions are required on them.

The central equation studied in the book is the nonlinear wave equation $u_{tt} - \Delta u = f(u_t, u_x, u_{tx}, u_{xx})$, where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ is the time, u_x denotes all the first spatial derivatives, etc., and Δ is the spatial Laplacian. (Direct dependence of f on u is omitted for technical reasons.) The "nonlinearity" f is an arbitrary C^∞ function whose minimum degree p among all its various arguments is given. That is, p is the order of vanishing of f where all its arguments vanish.

The simplest thing that can go wrong is that the solution can *blow up in a finite time*. For instance, the ode $dv/dt = v^2$ has the solutions $v = (C - t)^{-1}$. For any positive initial value C^{-1} at $t = 0$, the solution goes to $+\infty$ as $t \nearrow C$. For partial differential equations the same phenomenon can happen; for instance, the equation $u_t - \Delta u = u^2$ has precisely the same blowing-up solutions. They simply happen not to depend on x. But the partial differential equation also has lots of spatially dependent solutions, and some of them decay to zero as functions of time because of the dissipative property of the heat equation.

For the wave equation as well, some solutions decay to zero, as Shakespeare described in *Henry the Sixth* $(\mathrm{I}, 1, 2)$:

> Glory is like a circle in the water,
> Which never ceaseth to enlarge itself,
> Till by broad spreading it disperse to nought.

That is, a solution with finite energy spreads in $\mathbb{R}^n$. To guess the rate of spreading, we could argue that the energy of a solution of the linear wave equation is constant in time t and is closely related to the L^2 norm $\int |u|^2 \, dx$. A wave spreads at a constant rate, and hence by Huygens' Principle in time t it has spread within an approximate spherical ring of radius t and fixed width, which has volume $O(t^{n-1})$. If it is approximately uniform over this ring and the L^2 norm is approximately constant, its amplitude is $O(t^{-d})$, where $d = (n-1)/2$. This is the correct decay rate d, although our derivation of it has been entirely heuristic.

The basic theorem mentioned above for the nonlinear wave equation $u_{tt} - \Delta u = f(u_t, u_x, u_{tx}, u_{xx})$ is that if the initial data are $u(0, x) = \varepsilon\varphi(x)$, $u_t(0, x) = \varepsilon\psi(x)$, with φ and ψ being C^∞ functions with compact support, and if ε is sufficiently small, then there exists a global C^∞ solution. This theorem is valid for any C^∞

function f whose degree p at the origin (in all its variables) is big enough: $p > p^*$. The correct value for p^* comes from the interplay between the decay rate of linear spreading and the "peaking" effect of the nonlinearity. We determine the correct value of p^* by the following heuristic argument. A typical nonlinear term is $f(u_x, \dots) = |u_x|^p$, a pure power. An energy argument requires that the nonlinear term should have bounded L^2 norm. Now

$$\|(u_x)^p\|_2 \leq \|u_x\|_\infty^{p-1} \|u_x\|_2 \leq Ct^{-d(p-1)},$$

assuming the heuristic L^∞ decay rate. When the wave equation $u_{tt} - \Delta u = f$ is solved, the energy of u is bounded according to a classical formula by the time-integral of the L^2 norm of f. Thus, what we need is the integrability of $t^{-d(p-1)}$. That is, $d(p-1) > 1$, or

$$p > p^* \equiv 1 + 1/d \equiv 1 + 2/(n-1).$$

Thus, for instance, in three spatial dimensions the condition is $p > 2$, which unfortunately just misses quadratic nonlinearities. More on this case later.

If the global existence theorem fails because the degree is too small ($p \leq p^*$), then we expect that some solutions will blow up in a finite time T^*. We can regard T^* as a decreasing function of the size ε of the initial data. John aptly called T^* the *lifespan*. Then it is natural to estimate T^*. The same kind of heuristic argument as above leads to the correct estimate. Indeed, the equation has the form $Lu = \|u_x\|^p$ with L a linear operator so that $v = \varepsilon^{-1}u$ satisfies the equation $Lv = \varepsilon^{p-1}\|v_x\|^p = g$ and initial data independent of ε. Thus, as above, the L^2 norm satisfies $\|g\|_2 \leq C\varepsilon^{p-1}t^{-d(p-1)}$. After integrating in t, we want this norm to be bounded. Thus we get the condition $\varepsilon^{p-1}t^{1-d(p-1)} = O(1)$, or

$$T^* \sim C\varepsilon^{-k} \quad \text{where } k = ((p-1)^{-1} - d)^{-1}.$$

For instance, for dimension $n = 2$ and degree $p = 2$, we have $T^* \sim C\varepsilon^{-2}$. The important case $n = 3$, $p = 2$ is borderline; and although John proved blow up, the lifespan is very long: it turns out that $T^* \geq C\exp(1/\varepsilon)$.

Although various methods of proof have been given for these various theorems, some based on rather tricky estimates and on Nash-Moser techniques, it has turned out that they can be carried out most efficiently using the simple contraction principle! The space in which the contraction mapping acts is a Sobolev space based on a combination of L^2 and L^∞ norms with a sufficient number of derivatives. One way to obtain the optimal results is by replacing the usual derivative operators by the Lorentz-invariant operators of momentum, angular momentum, and space-time dilation.

The reason that $p = 2$ is so important is that it is the first nonlinear power in the Taylor expansion of f and therefore occurs most naturally. Klainerman observed that, in the critical three-dimensional case ($p = 1 + 1/d$), there is an important class of nonlinearities for which all the small initial data nevertheless lead to global solutions. These nonlinearities satisfy a natural structure condition, the *null condition*, that is related to the Lorentz invariance of the linear equation. This idea is an important ingredient in Christodoulou and Klainerman's famous construction of solutions of the Einstein equations of general relativity in a neighborhood of the flat Minkowski metric [1].

If f depends explicitly on u, the above heuristics are not applicable (because the energy depends on the derivatives of u and not on u itself). There is a remarkable theorem of John [4] that for $u_{tt} - \Delta u = |u|^p$ in three dimensions the critical power is exactly $p^* = 1 + \sqrt{2}$.

For $u_{tt} - \Delta u + u = f(u, u_t, u_x, u_{tx}, u_{xx})$, the so-called nonlinear Klein-Gordon equation, the basic decay rate is a little larger: $d = n/2$. Therefore, the condition of criticality is $p > 1 + 1/d = 1 + 2/n$. This includes the important case $p = 2$, $n = 3$. This global existence theorem was first proved by Shatah and by Klainerman and Ponce in the early 1980s.

The first results of this type for the small solutions of nonlinear wave equations were given by Segal [5] in 1968. The first proofs that used the simple contraction principle were by the reviewer in 1974 and for more general nonlinearities by Shatah [6] in 1982. The role of Lorentz inversion was first observed by Morawetz in the 1960s, and the key operators like angular momentum, dilation, and inversion were used explicitly in energy estimates by the reviewer in the 1970s and were used to construct the invariant norms by Klainerman in the 1980s. John's striking paper [4] stimulated other major contributions by John himself, Glassey, Klainerman, Shatah, Christodoulou, Ponce, Sideris, Pecher, Hörmander, and, more recently, Kovalyov, Lindblad, Li, and others. References up to 1989 may be found in [7].

Now to the book by Li and Chen. It is a nice exposition of the existence of small global solutions for three types of equations with very general nonlinearities. There are three chapters. The most important chapter treats the nonlinear wave equations discussed above. It uses the contraction idea in conjunction with Klainerman's 1985 method of invariant norms and the authors' improvements. A major theme in the book is the unification of the analysis of the global solutions (if $p > p^*$) and of the lifespan (if $p \leq p^*$). This is done elegantly using the contraction principle just the way it should be!

The history, however, is not accurately explained in this book. In particular, the early work of Segal and others is not mentioned, nor are the origins of the ideas on Lorentz invariance explained. Although the use of the simple contraction idea on a combination of L^2 and L^∞ spaces is presented by the authors as their own concept, it actually goes back in this very same context to the 1970s. Omitted from the book are the null condition, nonlinearities $f(u)$ independent of the derivatives, noninteger values of the degree p, and the Klein-Gordon case.

The first chapter addresses nonlinear heat equations $u_t - \Delta u = f(u, u_x, u_{xx})$. This is a good choice because it is a technically much simpler equation. The basic L^∞ decay rate $d = n/2$ comes directly from the classical heat kernel. With this value of d, the critical $p^* = 1 + 2/n$ and the lower bounds on the lifespan T^* are determined for the same reasons as above. The book carries out the existence theory using the contraction idea. It mentions that the key observation of blow up goes back to Fujita [2]. However, it barely mentions the huge literature on existence theory for parabolic equations.

The third chapter discusses nonlinear Schrödinger equations in a similar way, assuming some structure condition on the nonlinearity. Here again, $d = n/2$, so that we ought to have $p^* = 1 + 2/n$. However, the authors limit themselves to proving a nonoptimal theorem with a more restrictive condition on p. They should have indicated that there is gap between their condition and the known blow up criteria [3]. In fact, there are some recent existence results in this gap.

This book is a very nice exposition of an important set of results. It is exceptionally well organized, even though there is no index and a few misprints. One can learn many useful techniques that occur in nonlinear partial differential equations, including the treatment of fully nonlinear problems, Galerkin's approximation method, weak convergence techniques, Sobolev estimates on composite functions, Lorentz invariant norms, decay estimates, and, of course, sophisticated uses of the simple contraction principle. Not included are discussions of the null condition, of blow up (except briefly), of general solutions (i.e., large ones, which is another whole story!), or of boundary problems. Nevertheless, it would make a nice choice for a part of a graduate course designed as a sequel to a course on linear partial differential equations.

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Walter A. Strauss
Brown University
E-mail address: wstrauss@gauss.math.brown.edu