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BOOK REVIEW

Generalized hypergeometric functions, by Bernard Dwork. Clarendon Press, Oxford, 1991, 188 pp. $63.00. ISBN 0-19-853565-8.

Estimating exponential sums is a fundamental problem in the theory of Diophantine equations. A typical situation arises when one applies the Hardy-Littlewood circle method to estimate the number of integral solutions of an equation. For example, Kloosterman [12] investigated the exponential sums that now bear his name in order to determine which diagonal quaternary quadratic forms represent all sufficiently large positive integers. New opportunities arose with Deligne's proof of the Riemann hypothesis for varieties over finite fields and the resulting improvement in estimates for exponential sums of several variables [2]. Heath-Brown [9] used Deligne's work to show that every nonsingular rational cubic form in at least ten variables represents zero rationally. Hooley has applied extensions of Deligne's work, due to Katz and Milne, to a variety of Diophantine problems (see [10] and the references listed therein).

The sharpest bounds for exponential sums in several variables rest ultimately on Deligne's theorem [3]. To obtain useful bounds from Deligne's theorem, however, one first needs rather precise information about the l-adic cohomology groups associated to the exponential sum. The "best" situation is when all cohomology groups, except the middle-dimensional one, vanish, but finding simple conditions on the exponential sum that guarantee that this happens is a challenging problem and is still an active area of research [11, 1, 4]. For this reason, it is useful to investigate other cohomology theories associated to exponential sums, particularly, those that are especially amenable to direct computations of cohomology. In addition, other theories may reveal properties of exponential sums that are not apparent from the l-adic point of view, even in cases where the l-adic theory is well understood.

In this research monograph, the author applies his p-adic cohomology theory to the study of a general class of character sums over finite fields of characteristic p. His main concern is to study how sums vary within a family, which may depend on one or several parameters. The cohomology, it turns out, varies (p-adically) analytically with the parameters and can be described by systems of differential equations (or $\mathcal{D}$-modules) of hypergeometric type. In fact, the systems that arise are exactly those that are often referred to as $\mathcal{A}$-hypergeometric systems [6].

The main topics of the book are construction of p-adic cohomology spaces, determination of the region of analyticity of cohomology (as function of the parameters), and the study of the differential equations that arise. Various classical hypergeometric functions are discussed as examples of the general theory. We now make precise some of this general outline.

Let k be a finite field of cardinality q and characteristic p, and let $f^{(1)}, \dots, f^{(n_2)} \in k(\lambda_1, \dots, \lambda_m)[x_1, \dots, x_{n_1}]$. (In the book under review it is assumed that these polynomials are homogeneous in the x's. We believe this restriction to be unnecessary, but it results in no loss of generality.) We think of the f's as polynomials in the x's whose coefficients depend on some parameters $\lambda_1, \dots, \lambda_m$ that will take values in $\overline{k}$, the algebraic closure of k. Let $\chi_1, \dots, \chi_{n_1+n_2} \colon k^\times \to \mathbf{C}_p^\times$ be multiplicative characters, where $\mathbf{C}_p$ denotes the completion of the algebraic closure of the p-adic numbers $\mathbf{Q}_p$. There is a canonical multiplicative character $\omega : \overline{k}^\times \to \mathbf{C}_p^\times$, the Teichmüller character, characterized by the fact that its composition with reduction $\operatorname{mod} p$ is the identity on $\overline{k}^\times$. It generates the character group of every finite extension of k; hence, one may write $\chi_i = \omega^{-a_i}$, $0 \le a_i < q-1$. Write λ for $(\lambda_1, \dots, \lambda_m)$ and a for $(a_1, \dots, a_{n_1+n_2})$. Define a character sum

$$S_1(a/(q-1), \lambda) = \sum_{x_1, \dots, x_{n_1} \in k(\lambda)^\times} \prod_{i=1}^{n_1} \chi_i(x_i) \prod_{j=1}^{n_2} \chi_{n_1+j}(f^{(j)}(x)).$$

One defines $S_r(a/(q-1), \lambda)$ analogously, replacing $k(\lambda)$ by $k(\lambda)_r$, the extension of $k(\lambda)$ of degree r, and replacing each multiplicative character by its composition with the norm map from $k(\lambda)_r$ to $k(\lambda)$. Associated to this data is the L-function

$$L(a/(q-1), \lambda; t) = \exp\left(\sum_{r=1}^{\infty} S_r(a/(q-1), \lambda) \frac{t^r}{r} \right).$$

When the polynomials $f^{(j)}$ satisfy a certain regularity condition (which guarantees there is a single nonvanishing cohomology group), it is shown that there exists a p-adic matrix, analytic in $a/(q-1)$ and $\omega(\lambda)$, which determines $L(a/(q-1), \lambda; t)$. We make this more precise, while omitting certain technical hypotheses.

For $\mu \in \mathbf{Z}^{n_1+n_2}$ there is a square matrix

$$\gamma_\mu(A, \Lambda) (= \gamma_\mu(A_1, \dots, A_{n_1+n_2}, \Lambda_1, \dots, \Lambda_m))$$

whose entries are analytic functions of (A, Λ) on a certain region of $(\mathbf{C}_p)^{n_1+n_2+m}$. Let φ_μ be defined by $\varphi_\mu(A) = (A+\mu)/p$. Suppose $\lambda \in (\overline{k}^\times)^m$ satisfies $\lambda_i^{p^s} = \lambda_i$ for $i = 1, \dots, m$, and suppose we have a sequence $\mu^{(1)}, \dots, \mu^{(s)} \in \mathbf{Z}^{n_1+n_2}$ satisfying

$$\varphi_{\mu^{(s)}} \circ \cdots \circ \varphi_{\mu^{(1)}}(a/(q-1)) = a/(q-1).$$

Put

$$a^{(i)}/(q-1) = \varphi_{\mu^{(i-1)}} \circ \cdots \circ \varphi_{\mu^{(1)}}(a/(q-1)).$$

Then

$$L(a/(q-1), \lambda; t)^{(-1)^{n_1+n_2-1}} = \det\left(I - \left(\prod_{i=1}^{s} \gamma_{\mu^{(i)}}(a^{(i)}/(q-1), \omega(\lambda)^{p^{i-1}}) \right) t \right).$$

For gauss sums, an analogous formula holds. Since in that case the L-function is a linear polynomial, the corresponding matrix $\gamma_\mu(A, \Lambda)$ is one-by-one. Appropriately normalized, and for an appropriate choice of μ and Λ, it becomes the p-adic gamma function $\Gamma_p(A)$. Thus this formula may be regarded as extending the Gross-Koblitz formula for gauss sums [8].

The early chapters of the book are devoted to proving analytic properties of $\gamma_\mu(A,\Lambda)$. This matrix is constructed as the matrix of a certain mapping (Frobenius)

$$\alpha^*_{A,A',\Lambda} \colon \mathcal{K}'_{A',\Lambda^p} \to \mathcal{K}'_{A,\Lambda},$$

where $\mathcal{K}'_{A,\Lambda}$ is a family of finite-dimensional $\mathbf{C}_p$-vector spaces parametrized by A, Λ. Treating Λ as an indeterminate, one can regard $\mathcal{K}'_{A,\Lambda}$ as a module over the ring of differential operators in $\partial/\partial\Lambda_1, \ldots, \partial/\partial\Lambda_m$. Away from points Λ whose reduction $\bmod p$ gives a nonregular $f^{(j)}$, this defines a connection and, hence, gives rise to a first-order system of differential equations for each $\partial/\partial\Lambda_j$. Most of the remainder of the book is devoted to the study of these systems: solutions at ordinary and singular points, deformation mapping, contiguity relations, action of Frobenius on solutions.

One question that arises, particularly when one wants to write down interesting examples, is how does one explicitly describe the correspondence between character sums and differential equations? One way is to make a formal correspondence between integral formulas and character sums. Suppose, for example, $F(\Lambda)$ is a convergent power series at $(0,\ldots,0) \in \mathbf{C}^m$ with rational coefficients which has an integral formula

$$F(\Lambda) = \int_C x_1^{\alpha_1} \cdots x_{n_1}^{\alpha_{n_1}} (f^{(1)}(x))^{\beta_1} \cdots (f^{(n_2)}(x))^{\beta_{n_2}}\, d\frac{dx_1}{x_1} \cdots \frac{dx_{n_1}}{x_{n_1}}$$

for some domain C in $\mathbf{C}^{n_1+n_2}$, where the coefficients of the $f^{(i)}$'s are rational functions of Λ. Then for an appropriate choice of multiplicative characters, the system of p-adic differential equations corresponding to the character sum

$$\sum_{x \in (k(\lambda)^\times)^{n_1}} \prod_{i=1}^{n_1} \chi_i(x_i) \prod_{j=1}^{n_2} \chi_{n_1+j}(f^{(j)}(x))$$

has as a solution the series $F(\Lambda)$, where now the coefficients of the series are thought of as lying in $\mathbf{C}_p$. (A complete solution to this correspondence problem, which does not require knowledge of an integral formula for the power series $F(\Lambda)$, is now known. See [5].) Using this idea, the author discusses in Chapter 12 the character sums corresponding to various classical hypergeometric functions.

The last four chapters treat duality, i.e., functional equation for the L-function $L(a/(q-1), \lambda; t)$, for a variety of cases. The main condition is a restriction on $a/(q-1)$, called *nonresonance* in the terminology of [7]. This can be shown to imply purity of the corresponding character sum (i.e., all reciprocal roots of the L-function have absolute value $\sqrt{q}^{\,n_1}$).

Since the publication of this book, it has been discovered that there are important connections between Dwork's theory of hypergeometric functions and the theory of $\mathcal{A}$-hypergeometric functions of Gelfand, Graev, Kapranov, and Zelevinsky. It seems likely that these connections will stimulate further developments in both theories.

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