References

1. A Adolphson and S. Sperber, *Exponential sums and Newton polyhedra: cohomology and estimates*, Ann. of Math. (2) **130** (1989), 367–406.
2. P. Deligne, *La conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. **48** (1974), 273–308.
3. ———, *La conjecture de Weil*. II, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252.
4. J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra*, Invent. Math. **106** (1991), 275–294.
5. B. Dwork and F. Loeser, *Hypergeometric series*, Japanese J. Math. **19** (1993).
6. I. M. Gelfand, A. V. Zelevinskii, and M. M. Kapranov, *Hypergeometric functions and toral manifolds*, Funct. Anal. Appl. **23** (1989), 94–106.
7. I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Generalized Euler integrals and $\mathscr{A}$-hypergeometric functions*, Adv. Math. **84** (1990), 255–271.
8. B. Gross and N. Koblitz, *Gauss sums and the p-adic Γ-function*, Ann. of Math. (2) **109** (1979), 569–581.
9. D. R. Heath-Brown, *Cubic forms in ten variables*, Proc. London Math. Soc. (3) **47** (1983), 225–257.
10. C. Hooley, *On nonary cubic forms*, J. Reine Angew. Math. **386** (1988), 32–98.
11. N. Katz, *Sommes exponentielles*, Astérisque **79** (1980), 1–209.
12. H. D. Kloosterman, *On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$*, Acta Math. **49** (1926), 407–464.

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Integral inequalities and applications, by Drumi Bainov and Pavel Simeonov (translated by R. A. M. Hoksbergen). Kluwer Academic Publishers, Dordrecht, Boston and London, 1992, xi+245 pp., $110.00. ISBN 0-7923-1714-9

The history of integral equations begins with Abel's famous papers [1, 2] and continues with numerous contributions culminated at the turn of century in the works of Vito Volterra and Ivar Fredholm. Hence, it may appear somewhat surprising that only in 1919 Gronwall [4] discovered the integral inequalities and their use in studying problems of the theory of ordinary differential equations. During the last forty years the theory of integral inequalities has known a spectacular growth, and a number of monographs have been dedicated to this subject (see [3, 5, 6, 9, 11], partially or entirely). The list of references to this book contains 250 items. To the best of our knowledge, most of the significant contributions have been included in the references. Applications of integral inequalities are also treated with particular regard to the theory of ODE, PDE, IE, IDE, and FDE. These applications illustrate to a reasonable extent the role of integral inequalities in obtaining for the above-mentioned equations such properties as uniqueness of solution, dependence of the solution with respect to data, global existence, stability of solutions, error estimates in approximation procedures, and other types of estimates.

Let us point out the following important feature in dealing with integral inequalities. Assume, for instance, that we are given the integral inequality

$$\text{(I)} \qquad x(t) \leq a(t) + \int_0^t b(t, s, x(s))\, ds, \qquad t \in [0, T),$$

in which $a(t)$ and $b(t, s, x)$ are known. To solve (I) does not mean that we are looking for the class of those $x(t)$ satisfying (I) on some interval $[0, T_1) \subset [0, T)$. All we look for is an upper bound, say $\tilde{x}(t)$, for all solutions of (I), i.e., $x(t) \leq \tilde{x}(t)$ for any $x(t)$ satisfying (I).

The first chapter of the book is concerned with Gronwall-type inequalities, all of which are special cases of the linear inequality obtained from (I) for $b(t, s, x) = k(t, s)x$. For $k(t, s) \geq 0$ one obtains $x(t) \leq y(t)$, $t \in [0, T)$, where $y(t)$ is the (unique) solution of the linear equation $y(t) = a(t) + \int_0^t k(t, s)y(s)\, ds$.

The second chapter deals with nonlinear integral inequalities, including those known as Bihari's, Lakshmikantham's, and others'. These inequalities are basically of type (I). A key condition is the monotonicity property of the function b in x. The comparison method is developed in this chapter, including the case of scalar first-order differential inequalities $x'(t) \leq F(t, x(t))$.

The last chapter of the book is dedicated to various generalizations of the inequalities discussed in the first two chapters—namely, inequalities involving several integrals, integrodifferential inequalities, integral inequalities with delay, inequalities with several variables, and systems of integral inequalities.

There are a few items missing from the list of references. First, [10] by Tsalyuk contains a very general result of comparison type for scalar inequalities with Volterra operators (causal or nonanticipative). These inequalities are of the form ($\mathfrak{F}$) $x(t) \leq (Vx)(t)$, $t \in [0, T)$, where V stands for a Volterra operator, completely continuous and nondecreasing in x. If there exist two continuous functions $z_1(x) \leq z_2(x)$, $x \in [0, T)$, such that $z_1(t) \leq (Vz_1)(t)$, $z_2(t) \geq (Vz_2)(t)$ on $[0, T_1) \in [0, T)$, then the equation $x(t) = (Vx)(t)$ has minimal and maximal solutions, $x_{\mathrm{m}}(t)$ and $x_{\mathrm{M}}(t)$, such that $z_1(t) \leq x_{\mathrm{m}}(t) \leq x_{\mathrm{M}}(t) \leq z_2(t)$ on $[0, T_1]$. Under the extra conditions of uniqueness, one has $z_1(t) \leq x(t) \leq z_2(t)$, which means that subsolution, solution, and supersolution of $x(t) = (Vx)(t)$ are ordered as shown above. Tsalyuk [10] states that this result is a special case of a general result of Birkhoff and Tarski from the theory of partially ordered spaces.

The above result of Tsalyuk contains as special cases most of the results given by the authors in the first two chapters. The problem that remains to be solved in each particular case is getting the solution $x(t)$ or a supersolution of the equation $x(t) = (Vx)(t)$. That is what the authors (of the book or of the papers quoted in the book) have done.

In regard to the results included in Chapter III those in the first two sections also belong to the class of inequalities with abstract Volterra operator. Those related to integrodifferential equations or equations with delay belong to the class of functional differential inequalities of the form $\dot{x}(t) \leq (Vx)(t)$, with an abstract Volterra operator V. Such inequalities have been dealt with in [7] by

McNabb and Weir. This paper, with strong unifying role, does not appear in the authors' list of references.

The last section of Chapter III is dedicated to inequalities in partially ordered spaces. It is shown how to obtain from general results some of the usual results in the case of classical integral inequalities.

Another topic discussed in Chapter III is related to discrete inequalities such as $u_n \leq a_n + q_n \sum_{s=0}^{n-1} b_s u_s$, $n \geq 1$. Integral inequalities with impulses, mixing up the continuous and the discrete case (for instance, of the form $u(t) \leq a + \int_0^t b(s)u(s)\,ds + \sum_{t_k<t} \beta_k u(t_k))$, are also investigated. There are some integral inequalities in the literature which do not appear in this book. For instance, Pazy [8, p. 159] solves the inequality

$$\varphi(t, s) \leq A + B \int_s^t (t-\sigma)^{\alpha-1} \varphi(\sigma, s)\,d\sigma, \qquad 0 \leq s < t \leq \tau,$$

in which A, B, and α are positive constants. This inequality, which can be reduced to one of Gronwall type, has applications to semigroup theory, in connection with evolution equations.

The style of the text is rather concise but readable. This book will be useful to many readers working in Integral Equations, Integrodifferential Equations, Functional Differential Equations, and related areas.

A more systematic treatment of the topics covered in the book would likely result if the concept of abstract (causal) Volterra operator were used, instead of the classical Volterra operator. There is some overlapping with the material contained in other monographs dedicated to the subject [3, 5, 6, 9, 11], but the book has its own unity and structure.

References

1. N. H. Abel, *Solution de quelques problèmes à l'aide d'intégrales définies*, Mag. Naturvidenskab., Aargang I, bd. 2, 1823 (Oeurves Completes, T.1, B. 2, Christiania, 1881, pp. 11–27).
2. N. H. Abel, Auflösung einer mechanischer Aufgabe, Crelles J. (1826), 153–157.
3. P. R. Beesak, *Gronwall inequalities*, Carleton Math. Lecture Notes (1975), no. 11.
4. T. H. Gronwall, *Note on the derivative with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. **20** (1919), 292–296.
5. V. Lakshmikantham and S. Leela, *Differential and integral inequalities*, vol. I, Academic Press, New York, 1969.
6. A. A. Martynyuk and R. Gutowski, *Integral inequalities and stability of motion*, Naukova Dumka, Kiev, 1979. (Russian)
7. H. McNabb and G. Weir, *Comparison theorems for causal functional differential equations*, Proc. Amer. Math. Soc. **104** (1988), 449–452.
8. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York and Berlin, 1987.
9. J. Schröder, *Operator inequalities*, Academic Press, New York, 1980.
10. Z. B. Tsalyuk, *Volterra functional inequalities*, Yzv. Vyssh. Ucheb. Zaved. **3** (1969), 86–95. (Russian)
11. W. Walter, *Differential and integral inequalities*, Springer-Verlag, Berlin, 1979.

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