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Manifolds with singularities and the Adams-Novikov spectral sequence, by Boris I. Botvinnik. Cambridge University Press, London and New York, 1992, xv+181 pp., $37.95. ISBN 0-521-46208-1

From the preface: "The purpose of this book is to discuss some natural relations between geometric concepts of Cobordism Theory of manifolds with singularities and the Adams-Novikov spectral sequence." Before discussing the book itself, we will give some background and define the terms used in its opening sentence.

Cobordism theory of manifolds (without singularities) was one of the great successes of algebraic topology in the 1950s and 1960s. The basic idea is the following: Two closed (i.e., compact, smooth, and without boundary) n-dimensional manifolds, M_1 and M_2, are said to be *cobordant* if there is a smooth $(n+1)$-dimensional manifold W whose boundary is the disjoint union of M_1 and M_2.

In particular, M_2 could be empty, in which case we are requiring M_1 to be the boundary of some W. The most easily visualized closed manifolds, namely, the circle and the closed oriented surfaces, are boundaries. The simplest closed manifold that is not a boundary is the projective plane $\mathbf{R}P^2$, the space of lines through the origin in $\mathbf{R}^3$. It is not easy to visualize, since it cannot be embedded in $\mathbf{R}^3$.

Corbordism (being cobordant) is easily seen to be an equivalence relation, so one would like to classify closed n-manifolds up to corbordism. The set of equivalence classes $\mathcal{N}_n$ is easily seen to be an Abelian group under disjoint union. One also has a homomorphism $\mathcal{N}_m \otimes \mathcal{N}_n \to \mathcal{N}_{m+n}$ induced by Cartesian product. This makes the direct sum of all of the $\mathcal{N}_n$ a graded ring denoted by $\mathcal{N}_*$, the *cobordism ring.*

At first glance the determination of the structure of this ring appears to be hopeless. Where does one start? This is the subject of some remarkable work in the 1950s by René Thom, for which he won the Fields Medal in 1958. (The original source is Thom's paper [Tho]; the account by Milnor-Stasheff [MS] is highly recommended.) We do not have the space to describe this construction here. The upshot is that the group $\mathcal{N}_n$ is isomorphic to the nth homotopy group of a certain topological space (or more precisely, a spectrum) denoted by MO. Thus Thom's theorem translates the geometric problem of describing the cobordism ring $\mathcal{N}_*$ into the homotopy theoretic problem of computing the homotopy groups $\pi_*(MO)$. Not only that, Thom was able to make the latter calculation and show that

$$\mathbf{Z}/(2)[x_2, x_4, x_5, x_6, \ldots]$$

where x_n denotes the cobordism class of a certain n-manifold for each positive integer n that is *not* one less than a power of two. In particular, x_2 is the class of the projective plane $\mathbf{R}P^2$.

One can alter the problem by requiring all manifolds in sight to possess some additional structure, such as an orientation or (suitably defined) spin, complex or symplectic structure. Thom's methods still apply, but we end up with a space other than MO. The appropriate spaces for the four structures above are called MSO, $MSpin$, MU, and MSp respectively. The homotopy groups (and hence the corresponding cobordism rings) of the first three were computed by various authors in the 1960s. An account of this work can be found in Stong's book [Sto]. However, the symplectic case, i.e., the computation of $\pi_*(MSp)$, has proved to be far more difficult. *It is the motivating problem for the book under review.*

The study of the complex case by Milnor [Mil] and Novikov [Nov1, Nov2] led to what might be called the American and Russian schools of cobordism theory. In the last twenty years the former has directed its efforts toward applying cobordism theory to problems in homotopy theory; this work led eventually to the nilpotence and periodicity theorems of Devinatz, Hopkins, and Smith [DHS]. The central technical tool in this enterprise has been the Adams-Novikov spectral sequence, introduced by Novikov in [Nov3] and described in the reviewer's book [Rav]. It is an algebraic tool for computing homotopy groups.

On the Russian side the emphasis was on classifying symplectic manifolds up to cobordism. Until recently there was little contact between the two schools for obvious reasons. Botvinnik's book will make the Russian school's work more accessible to western topologists, who have virtually ignored it up until now. The central idea here is that *the Adams-Novikov spectral sequence, when* *applied to MSp, has a rich geometric interpretation in terms of "manifolds with singularities".*

What are "manifolds with singularities"? They are actually generalizations of manifolds rather than manifolds with additional structure, as the term implies. For the simplest example of such, fix a closed k-dimensional manifold P_1. Let cP_1 denote the cone on P_1, i.e., the topological quotient of the cylinder $[0, 1]\times P_1$ obtained by collapsing $\{0\}\times P_1$ to a single point. For $n > k$ a *closed n-dimensional manifold M with singularity of type P_1* is, roughly speaking, a topological space in which each point has a neighborhood that is homeomorphic either to $\mathbf{R}^n$ (the usual requirement for a manifold) or to $cP_1 \times \mathbf{R}^{n-k-1}$, subject to the usual smoothness conditions. The set of points having the unusual type of neighborhood is necessarily a closed manifold V of dimension $n - k - 1$. We also require that V has a neighborhood homeomorphic to $V \times cP_1$. The complement of the interior of this neighborhood is an ordinary manifold N with boundary diffeomorphic to $V \times P_1$. Alternatively, we could define a manifold with singularity of type P_1 to be an ordinary manifold N with boundary diffeomorphic to $V \times P_1$ for some V.

One can generalize this definition to singular manifolds with boundary, in which the singular locus V has a boundary. This enables one to define cobordism of such objects. Another generalization is to allow V itself to be a manifold with singularity of type P_2. The resulting object is called a manifold with singularity of type (P_1, P_2). Proceeding inductively, one defines manifolds with singularities of type $\Sigma = (P_1, P_2, \dots)$ for a suitable sequence of manifolds Σ. There are long exact sequences relating the cobordism groups for various sequences related to a given Σ. Naturally this requires some tricky bookkeeping.

Eventually one arrives at the Σ-singularities spectral sequence, which in principle enables one to compute the nonsingular cobordism groups in terms of the singular ones. (This is useful because in certain cases the homotopy theoretic problem associated with the singular cobordism groups is more accessible than that associated with the nonsingular ones.) This spectral sequence is the subject of Botvinnik's Chapter 1. Its multiplicative properties, which are even trickier, are developed in Chapter 2.

The theory developed in the first two chapters is quite general and very geometric. There are over twenty drawings to assist the reader in visualizing the constructions. This much explicit geometry is rarely seen in contemporary algebraic topology.

The actual aim of this theory does not emerge until the third chapter. It begins by setting up the algebraic machinery associated with the Adams-Novikov spectral sequence for computing $\pi_*(MSp)$, the symplectic cobordism ring. Then we learn a remarkable fact: *there is a sequence Σ of symplectic manifolds* (constructed twenty years ago by Nigel Ray) *such that the* (*geometrically constructed*) *Σ-singularities spectral sequence coincides with the* (*algebraically constructed*) *Adams-Novikov spectral sequence.*

In more technical language, the methods of this book lead to a very efficient E_1-term for the Adams-Novikov spectral sequence. In the final chapter its multiplicative properties are established and the first differential d_1 is determined. This, in turn, gives a very explicit description of the E_2-term for a theory that is just one singularity away from MSp.

Given the wide range of technical tools that are introduced, this book is remarkably focused. For example, the theory of two-valued formal groups laws, developed at great length by the Russian school in the 1970s to study symplectic cobordism, is described here in just three pages, giving precisely what is needed and nothing more.

Botvinnik describes the state of the art of symplectic cobordism theory in a very coherent way. The problem is far from being solved, but a major watershed has been reached. This book will be invaluable for algebraic topologists interested in the achievements of the Russian school.

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Global properties of linear ordinary differential equations, by Frantisek Neuman. Mathematics and its Applications, vol. 52, Kluwer Academic Publishers, Dordrecht, 1991, xv+320 pp., $129.00. ISBN 0-7923-1269-4

The linear ordinary differential equation

$$Py \equiv y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0 \tag{1}$$

on $I = (a, b)$ with $-\infty \le a < b \le \infty$ can be solved when $n = 1$ in closed form $y = c\exp(-\int p_0)$. For $n > 1$ there is no "closed form" solution available, not even in the case of

$$y'' + qy = 0. \tag{2}$$