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BOOK REVIEW

Global properties of linear ordinary differential equations, by Frantisek Neuman. Mathematics and its Applications, vol. 52, Kluwer Academic Publishers, Dordrecht, 1991, xv+320 pp., \$129.00. ISBN 0-7923-1269-4

The linear ordinary differential equation

$$Py \equiv y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0 \tag{1}$$

on $I = (a, b)$ with $-\infty \le a < b \le \infty$ can be solved when $n = 1$ in closed form$y = c\exp(-\int p_0)$. For $n > 1$ there is no "closed form" solution available, not even in the case of

$$y'' + qy = 0\,. \tag{2}$$

So it is perhaps not surprising that many of the most basic questions about equation (2) are still waiting for complete answers: Are the solutions oscillatory? bounded? in $L^2(I)$? Do they have limit 0 at an endpoint? Do they grow or decay exponentially? More generally, what is their asymptotic behavior at a singular endpoint?

This is the current state of affairs in spite of the fact that equation (2) is the most widely studied differential equation in the world. Thousands of papers have been written about this equation not only by mathematicians but by others as well. In 1836 Sturm and Liouville realized that, since no "formula" for solutions of (2) was available for all coefficients q, properties of such solutions would have to be obtained from the equation itself. Thus started a vast enterprise on what today we call Sturm-Liouville problems based on equation (2) or the more general equation

$$(py')' + qy = 0\,. \tag{3}$$

Coefficient conditions

We restrict our discussion of these equations to intervals of the real line and to real-valued solutions for real-valued coefficients. By a solution of (1) we mean a real-valued function y such that $y^{(j)} \in AC_{\text{loc}}(a, b)$, $j = 0, \ldots, n-1$, and $Py = 0$ a.e. on I. Here $AC_{\text{loc}}(I)$ denotes the functions which are absolutely continuous on all compact subintervals of I. Thus we assume that p_j is real valued and satisfies

$$p_j \in L_{\text{loc}}(a, b), \qquad j = 0, 1, , \ldots, n-1, \tag{4}$$

where $L_{\mathrm{loc}}(a,b)$ denotes the set of functions on (a,b) which are Lebesgue integrable on each compact subinterval. The conditions (4) are necessary and sufficient for all initial value problems with the initial conditions specified at any point $c \in (a,b)$ to have unique solutions. The proof of the sufficiency is well known and can be found in any "good" differential equations book, for example, [10]. On the other hand, the necessity, although easy to prove, seems not to be widely known; see [3]. Thus conditions (4) are natural "minimal" assumptions on the coefficients of (1). Most elementary differential equations texts, many advanced texts including the book under review, and quite a few authors of recent research papers, particularly in oscillation theory, assume that the coefficients are continuous. In most cases this assumption is not needed and much too strong. For instance, it rules out piecewise constants which yield a rich source of examples. There is even a "conjecture" that any phenomenon occurring for equation (1) is exhibited by an equation of the same order with piecewise constant coefficients. The state-of-the-art computer code "SLEDGE" [5] used to compute eigenvalues of Sturm-Liouville problems is based on piecewise constant approximations of the coefficients.

Oscillation

Consider equation (2) on $I = [0,\infty)$ with $q \in L_{\mathrm{loc}}(I)$. It follows from uniqueness that the zeros of any nontrivial solution of (2) in I are isolated. Hence they can accumulate only at ∞. Such a solution is called oscillatory (at ∞). By the Sturm separation theorem if one nontrivial solution is oscillatory, then they all are. Thus oscillation is a property of the equation itself, and the equations (2) fall into two mutually exclusive classes: oscillatory (O) and nonoscillatory (NO). How can one tell directly from the coefficient q whether or not the equation is oscillatory? From the Sturm comparison theorem it follows that

$$q(t) \geq \varepsilon, \qquad \varepsilon > 0 \Rightarrow \mathrm{O}, \tag{5}$$

and

$$q(t) \leq 0 \Rightarrow \mathrm{NO}\,. \tag{6}$$

Comparison with the Euler equation and its refinements yields a sequence of successively stronger sufficient conditions for oscillation:

$$q(t) \geq \frac{1+\varepsilon}{4t^2}, \qquad \varepsilon > 0 \Rightarrow \mathrm{O}; \tag{7}$$

$$q(t) \geq \left(\frac{1}{4t^2} + \frac{1+\varepsilon}{4(t\log t)^2}\right), \qquad \varepsilon > 0 \Rightarrow \mathrm{O}; \tag{8}$$

$$q(t) \geq \left(\frac{1}{4t^2} + \frac{1}{4(t\log t)^2} + \frac{1+\varepsilon}{4(t\log\log t)^2}\right), \qquad \varepsilon > 0 \Rightarrow \mathrm{O}; \tag{9}$$

and so on. Each of these conditions is sharp. This follows from the fact that for q equal to the function on the right with $\varepsilon = 0$, the solutions can be computed explicitly and are seen to be NO.

Thus it is clear that the "border" between oscillation and nonoscillation is nebulous. That it is even more slippery than these results and examples indicate can be seen from the telescoping principle of Kwong and Zettl [6, 7]. It is based on the observation that zeros of nontrivial solutions of (2) depend on the local—not global—behavior of the coefficient q. Hence natural oscillation conditions are of "interval" type; i.e., they limit the behavior of q only on a sequence of intervals going to ∞ with only very mild restrictions on the complementary intervals to keep the problem regular there.

Start with any known oscillatory equation (2), and choose a sequence of points $a_k \to \infty$. Cut the plane at each vertical line $t = a_k$, and pull the two half planes apart forming a gap of arbitrary finite length. Now fill the gap with any piecewise continuous function whose integral over the gap is nonnegative. (In particular, the gap can be filled with the zero function.) This is stronger than what is required to keep the problem regular, but it is needed for the proof in [6]. Do this at each point a_k. Equation (2) with this new coefficient is also oscillatory.

The telescoping principle can be used to produce new oscillatory equations from those already known, including those near the "border" between oscillation and nonoscillation, and also to extend known sufficient conditions, be they pointwise or integral conditions, to more general conditions of "interval" type.

Implicit in the above discussion is the fact that sufficient conditions for nonoscillation cannot be of interval type since the coefficient can produce zeros on the complementary intervals. This gives some insight into why verifiable, necessary, and sufficient conditions for oscillation are so difficult to obtain: they must be of interval type and at the same time not of interval type.

Another illustration of the delicate dependence of oscillation on the coefficients is provided by a form of the Mathieu equation:

$$y'' + (\lambda - \sin(t))y = 0 \quad \text{on } [0, \infty)\,. \tag{10}$$

It is clear from the Comparison Theorem that this equation is oscillatory for $\lambda > 1$ and nonoscillatory when $\lambda \leq -1$. Furthermore, there is a number σ_0 such that equation (10) is O for $\lambda > \sigma_0$ and NO for $\lambda < \sigma_0$. (This oscillation constant σ_0 is the starting point of the essential spectrum of any selfadjoint realization of (10) in the Hilbert space $L^2(0, \infty)$.) What is σ_0? Estimates can be obtained in various ways (see [1]), and it can be shown that $\sigma_0 \cong -0.378$. What is its exact value? This is an open question.

The limit-point/limit-circle dichotomy

The situation with regard to limit-point (LP) and limit-circle (LC) conditions is entirely similar: "Natural" (LP) conditions are of interval type, while (LC) conditions are not. (Equation (2) with $q \in L_{\text{loc}}(I)$ and $I = [0, \infty)$ is LC on I if and only if all of its solutions are in $L^2(I)$; otherwise (2) is LP. The terms LC and LP stem from a construction used by Weyl in his celebrated 1910 paper; see [10].) From this perspective it is not surprising then that there are no known necessary and sufficient conditions, which can be verified in every instance, for (2) or (3) to be in the LP case.

The above remarks are intended to point out that there are classes of equations (2) for which it is not known if the solutions oscillate or are in $L^2(0, \infty)$ and to give the reader some insight as to why these basic questions remain unanswered in spite of the voluminous literature on these problems dating back more than 150 years in the case of oscillation and over 80 years on the LP/LC classification. Of course, much less is known for higher-order equations (1).

There are people who claim that in view of the many "strong" known sufficient conditions the second-order case is "solved" except for a few "pathological" cases. This claim is irrefutable since they can simply take all the unsolved cases to be pathological by definition! No knowledgeable person would make such a claim for the higher-order case.

One more point before we come to the specifics of the book under review: *Equation* (1) *does not encompass all linear ordinary nth order differential equations*. Besides the obvious point that the zeros of a nonunit leading coefficient may cause singularities in the interior of the underlying interval, there is a not-so-obvious and more important point. This can be illustrated with equation (3). For $p \in C^1$ equation (3) can be strong out into the form (1) but with leading coefficient p; however, this restriction on p is much too strong, unnecessary, and artificial. The basic existence and uniqueness theorem holds for (3) under just local integrability assumptions on q and $1/p$ [3]. Note also that p may change sign and may have "mild" zeros. Even for smooth p the form (3) has advantages over a strung out version of it into the form (1): it is symmetric (formally selfadjoint). Many of the celebrated equations of mathematical physics appear in the form (3).

The change of independent variable

$$x = \int_c^t 1/p \tag{11}$$

transforms (3) into (2) only for p satisfying some strong conditions; and even when these conditions are satisfied, the form (3) is often preferable to (2). We refer to the form (3) as a quasi-differential equation. In the higher-order case the relationship between (1) and the class of linear ordinary quasi-differential equations is much more complicated [4], but the latter class is larger than the former by many degrees of freedom.

Faced with a question about (1) which one cannot answer, a popular way to proceed is to transform (1) into a simpler form for which one can get an answer. The most widely used such transformations are the transformation of the independent variable $x = h(t)$, of which (11) is an example, and of the dependent variable $z(t) = f(t)y(t)$. Of course, these can be combined to form

$$z(t) = f(t)y(h(t))\,. \tag{12}$$

It is the study of this transformation (12) for equations of the form (1) with smooth coefficients that is the subject matter of the book under review. I hope the above remarks have convinced the reader that this is a worthwhile endeavor. When can two equations of the form (1) be transformed into each other? Are there effective criteria for determining this? (effective in the sense that one can decide on the basis of a simple and direct inspection of the coefficients). Chapter 4 is devoted to these questions. The first three chapters are preliminary, while in Chapter 5 the algebraic structure of the transformations which transform a given equation into itself is investigated. Chapter 7 is devoted to the question: Given an equivalence class of equations, is there a "natural" canonical representative? Chapter 8 studies invariants. The rest of the book is devoted to some applications and related results.

These include a rather interesting interpretation of oscillation in terms of certain curves on the sphere in $\mathbb{R}^n$.

The use of the transformation (12) in the study of linear ordinary equations dates back at least to the middle of the last century. This book provides a good up-to-date account. It makes accessible to a wide audience the work of the strong Czech and Slovak school, centered on Boruvka and its students and disciples, on ordinary linear differential equations. There are sixteen pages of references—some quite recent, others from the last century.

The organization is logical and well structured, but the English usage is poor, and the notation is unnecessarily complicated and awkward. Throughout the book the word "paragraph" is used to mean "section", definite articles are routinely omitted, and strangely worded phrases appear. A typical example is on page 18: "Each equation of set $\mathbf{A_n}$ can be written in the form

$$y^{(n)} + p_{n-1}y^{n-1} + \cdots + p_0 y = 0$$

with the unit-leading coefficient and $p_i \in C^0(I)$ for $i = 0, \ldots, n-1$. We denote the equation by $P_n(y, x; I)$ to express explicitly the dependent and independent variables, the interval of definition and the order of the equation. Also coefficient p_i of $y^{(i)}$ is denoted by the same, however small letter that is as a capital letter used for indication of equation $P_n(y, x; I) \ldots.$ We adopt notation

$$|P_n(y, x; I)| = |P_n| = |P|$$

for the... expression...."

The notation for the transformation (12), which is the subject of the book, is found on page 26:

$$\overline{\alpha} = \langle\langle f, h \rangle\rangle_P \, .$$

Why the bar over α? Why a slanted double inner product symbol? Also, what is the point of the vertical bars for a symbol denoting an expression, for example, $|P|$?

There is a tendency to dress up simple concepts in fancy language. Thus on pp. 46–47 we find: "Brandt groupoid of the Ehresmann groupoid, that means invariant with respect to the morphisms.... a structure of global transformations can be described in the frame of the theory of categories." "Stationary groups" for the heading of Chapter 6 may sound fancy but is not very informative. These fancy terms play only a peripheral role at best. "Change of variables for ordinary linear differential equations" would be a much more informative title for this book.

Since English is not the author's first language, it is not fair to criticize his use of it. It is fair to criticize the editors and the publisher.

Notwithstanding these quibbles about English, notation, and fancy terminology, the reviewer has a high opinion of this book. It is a valuable contribution to the literature. The author has provided an accessible, essentially self-contained, account of the change of variables transformation for ordinary linear differential equations in classical form with smooth coefficients. Anyone with a serious interest in this transformation can benefit from this book.

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