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HARMONIC ANALYSIS OF FRACTAL MEASURES INDUCED BY REPRESENTATIONS OF A CERTAIN C*-ALGEBRA

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Abstract. We describe a class of measurable subsets Ω in $\mathbb{R}^d$ such that $L^2(\Omega)$ has an orthogonal basis of frequencies $e_\lambda(x) = e^{i2\pi\lambda\cdot x}(x \in \Omega)$ indexed by $\lambda \in \Lambda \subset \mathbb{R}^d$. We show that such spectral pairs (Ω, Λ) have a self-similarity which may be used to generate associated fractal measures μ with Cantor set support. The Hilbert space $L^2(\mu)$ does not have a total set of orthogonal frequencies, but a harmonic analysis of μ may be built instead from a natural representation of the Cuntz C*- algebra which is constructed from a pair of lattices supporting the given spectral pair (Ω, Λ). We show conversely that such a pair may be reconstructed from a certain Cuntz-representation given to act on $L^2(\mu)$.

1. Introduction

Let Ω be a subset in d real dimensions (i.e., $\Omega \subset \mathbb{R}^d$, $d \geq 1$), and suppose that Ω has finite positive d-dimensional Lebesgue measure. Let $L^2(\Omega)$ be the corresponding Hilbert space with the usual inner product given by

$$\langle f, g\rangle = m_d(\Omega)^{-1} \int_\Omega \overline{f(x)}\ g(x)\ dx$$

where $dx := dx_1 \cdots dx_d$, and $m_d(\Omega)$ denoting the Lebesgue measure of Ω. Motivated by a problem of I. E. Segal and a paper by B. Fuglede [Fu], we considered in [JP1–3] the problem of deciding, for given Ω, when $L^2(\Omega)$ may possibly have an orthogonal basis of frequencies: For $\lambda \in \mathbb{R}^d$, let $x \cdot \lambda = \sum_{j=1}^d x_j\lambda_j$ be the usual dot product, and set

$$e_\lambda(x) = e^{i2\pi\ x\cdot\lambda}\ . \tag{1}$$

We say that two vector frequencies λ, λ' in $\mathbb{R}^d$ are *orthogonal* on Ω if

$$\int_\Omega e^{i2\pi(\lambda'-\lambda)\cdot x} dx = 0\ .$$

When Ω is further assumed open in $\mathbb{R}^d$, this problem is directly connected (see [Fu, JP1]) with the problem of finding simultaneous commuting selfadjoint extension operators for the partial derivatives $\sqrt{-1}\frac{\partial}{\partial x_j}$ $(1 \leq j \leq d)$ acting on $C_c^\infty(\Omega)$ (= all smooth compactly supported functions in Ω). In general, the problem may be given a group-theoretic formulation, and, in this form, we showed in [JP1] that it relates

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directly to a property of the representation ring generated by a certain induced representation. (See (2) below.)

2. Classical Examples

The most obvious examples of sets Ω with the basis property are measurable sets in $\mathbb{R}^d$ which are *fundamental domains* of lattices (see [Fu, JP1]). Let Γ be a rank d *lattice*, and let Γ^0 be the *dual lattice*.

$$(\text{Recall} \quad \Gamma^0 = \{\lambda \in \mathbb{R}^d : \lambda \cdot s \in \mathbb{Z} \ , \ \forall s \in \Gamma\}.)$$

Suppose Ω is a measurable fundamental domain for Γ. It is a simple matter to show then that $\{e_\lambda : \lambda \in \Gamma^0\}$ is an orthogonal basis for $L^2(\Omega)$. This elementary class of examples is in fact characterized by a multiplicative property (see [JP1, 2]), and they are called *multiplicative*. A pair—(Ω, Λ) such that $0 \in \Lambda$, and $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$—is called a *spectral pair*, and the set Λ is called the *spectrum*. We further showed in [JP1] that every spectral pair (Ω, Λ) in d dimensions may be factored, $(\Omega, \Lambda) \simeq (\Omega', \Lambda') \times (\Omega'', \Lambda'')$, such that the factors each are spectral pairs in dimensions d', d'' respectively, $d' + d'' = d$, (Ω', Λ') is multiplicative, and (Ω'', Λ'') is in "the other extreme". Specifically, this second factor generates a representation ring which is a copy of the algebra of all q by q complex matrices where q is a certain cover-multiplicity (see [JP2]), and (Ω'', Λ'') is called a *simple factor*.

3. Spectral pairs

In this paper, we shall consider the simple factors in more detail and show that they are associated with "fractals" in a sense which we proceed to describe. If (Ω, Λ) is a spectral pair in d dimensions, consider the group $K = \Lambda^0 = \{s \in \mathbb{R}^d : s \cdot \lambda \in \mathbb{Z}, \forall \lambda \in \Lambda\}$. We further showed in [JP1] that K is a rank d lattice and that there is a canonical embedding of Ω into the torus $\mathbb{R}^d/K$ such that the image Ω' of Ω on the torus again has the basis-property (relative to Haar measure on the torus) and the spectrum of Ω' is the same set Λ. We say that the pair (Ω', Λ) is in *reduced form*.

We have a second closed subgroup A in $\mathbb{R}^d$ directly associated with some given spectral pair (Ω, Λ),

$$A = \{a \in \mathbb{R}^d : x + a \in \Omega + \Lambda^0 \text{ (a.e.) } x \in \Omega\} \ .$$

Define a unitary representation U_t $(t \in \mathbb{R}^d)$, acting on $L^2(\Omega)$, given by

$$U_t e_\lambda = e^{i2\pi t\cdot\lambda} e_\lambda \qquad (t \in \mathbb{R}^d, \ \lambda \in \Lambda) \ , \tag{2}$$

and note that A may be characterized alternatively as the group

$$\{t \in \mathbb{R}^d : U_t \quad \text{acts multiplicatively on} \quad L^2(\Omega)\} \ .$$

When $t \in A$, then

$$U_t f(x) = f(x+t) \ , \quad \text{a.e. } \ x \in \Omega' \ , \ \forall f \in L^2(\Omega') \tag{3}$$

where the sum $x + t$ is in the torus $\mathbb{R}^d/K$. Hence, we get A acting as a group of torus-translations on Ω'.

FIGURE 1.

We say that some given spectral pair (Ω, Λ) is *multiplicative* if $A = \mathbb{R}^d$ and is a *simple factor* if A is a lattice in $\mathbb{R}^d$. There is a sense in which simple factors may be generated by lattice systems, but we do not yet have a complete structure theorem which covers all simple factors. It is not known if, for a simple factor with associated lattices K and A, the *degenerate* case $K = A$ may occur. (We expect not!) In [JP1], we proved the following result (which will be needed below) about nondegenerate simple factors:

Theorem 1 (see [JP1], Theorem 6.1]). *Let (Ω, Λ) be a spectral pair in $\mathbb{R}^d$, and suppose that the group S, given by $S = \{s \in \Lambda : s + \Lambda = \Lambda\}$, is a lattice. Let $\Gamma = S^0$, and suppose*

(i) *$A \subset \Gamma$, and*
(ii) *there is a section L for S in Λ such that A separates points on L (i.e., when $\ell, \ell' \in L$, $\ell \neq \ell'$, then there is some $a \in A$ s.t. $e^{i2\pi\ell\cdot a} \neq e^{i2\pi\ell'\cdot a}$).*

Then it follows that every measurable section D' inside Ω' (reduced form) for the action (3) of A by translation is a fundamental domain for Γ and, moreover, that

$$\Omega' = \bigcup_{a \in A/K} (D' + a) \tag{4}$$

and

$$(D' + a_1) \cap (D' + a_2) = \varnothing$$

for all $a_1 \neq a_2$ *in* A/K.

3.1. **Spectral duality.** In studying more general simple factors, we introduced in [JP3] an *inductive limit construction* which applies to the basic factors described in Theorem 1, and we found, as the limit object, the Hilbert space $L^2(\mu)$ where μ is a Hausdorff measure of fractional dimension (see [Fa, Hu, St1–3]). Such measures are known to be supported by Cantor type-sets, $\mathcal{C}$, say (see [Hu]), but typically the Lebesgue measure of $\mathcal{C}$ is zero. We now show that $\mathcal{C}$ may be built by self-similarity from simple factors.

3.2. Let (Ω, Λ) be a spectral pair subject to the conditions in Theorem 1; let $K \subset A \subset \Gamma$ be the associated lattices; let L be the section in Λ (assume $0 \in L$); and finally, let R be the inclusion matrix for $K \subset \Gamma$. (Let $\{u_i\}_{i=1}^d$ be generators for K over $\mathbb{Z}$ and $\{v_i\}_{i=1}^d$ for Γ; then $R \in M_d(\mathbb{Z}) \cap \mathrm{GL}_d(\mathbb{R})$ may be defined by $u_i = \sum_j R_{ij} v_j$. Recall $K = \{\sum_i n_i u_i \,:\, n_i \in \mathbb{Z}\ ,\ 1 \leq i \leq d\}$, and similarly for Γ.) Since $L \subset K^0$, we may consider affine mappings, $s \mapsto Rs + \ell$, acting on the lattice K^0. This map will be denoted τ_ℓ, and the underlying lattice K^0 will be understood from the context. Consider the mapping $\tau_0(s) = Rs$ given by matrix-multiplication. When the bases (u_i) for K and (v_i) for Γ are given, let (u_i^*) for K^0 and (v_i^*) for Γ^0 be dual bases, i.e., $u_i^* \cdot u_j = v_i^* \cdot v_j = \delta_{ij}$, $1 \leq i, j \leq d$. For $s = \sum_i s_i u_i^*$ with integral coordinates, $s_i \in \mathbb{Z}$, we have

$$\tau_0(s) = \sum_i (Rs)_i u_i^* = \sum_i s_i v_i^* \ ,$$

and $(Rs)_i = \sum_j R_{ij} s_j$. Note then that $\tau_0(K^0) \subset K^0$, and each τ_ℓ, $\ell \in L$, is affine on the lattice K^0. If Γ^0 is identified with a sublattice in K^0, then $\tau_0(K^0) = \Gamma^0$, and the matrix-transpose $R'_{ij} = R_{ji}$ *is* the inclusion-matrix for the dual lattice-inclusion $\Gamma^0 \subset K^0$.

3.3. **The Fractal Measure.** Also consider the affine maps S_b on $\mathbb{R}^d$ given by

$$S_b x = R^{-1} x + b\ , \qquad x \in \mathbb{R}^d\ . \tag{5}$$

In formula (5), the term $R^{-1}x$ is really $\tau_0^{-1}(x)$, which is to say that the matrix-product $R^{-1}x$ must refer to the same basis (u_i^*) (for K^0) that was used in calculating τ_0 above. (In some different basis, of course, the matrix will change, i.e., R becomes ARA^{-1} with A denoting the associated transform matrix.)

Let N be the cardinality of L; by Theorem 1, it is also the order of the group A/K. Pick a subset $\mathcal{B} \subset A$, $0 \in \mathcal{B}$, representing the elements in A/K, equivalently a section for the quotient; and let the affine maps S_b be indexed by $b \in \mathcal{B}$. By Hutchinson's theorem (see [Hu, St1–2]) there is self-similar probability measure μ on $\mathbb{R}^d$ such that $\mu = \frac{1}{N} \sum_{b \in \mathcal{B}} \mu \circ S_b^{-1}$, or, equivalently,

$$\int f(x)\, d\mu(x) = \frac{1}{N} \sum_{b \in \mathcal{B}} \int f(S_b x)\, d\mu(x)$$

for measurable functions f on $\mathbb{R}^d$. We show in [JP3] that there is a "Cantor set" $\mathcal{C} \subset \mathbb{R}^d$, which is built from iteration of the decomposition (4) and self-similarity and which supports μ, i.e., $\mu(\mathcal{C}) = 1$. We let $L^2(\mu)$ be the corresponding Hilbert space.

3.4. **The Cuntz Algebra.** Our two theorems below connect the classical harmonic analysis of (Ω, Λ) to the associated fractal measure μ:

Theorem 2. *Let (Ω, Λ) be a nondegenerate simple factor given by the conditions in Theorem 1 with matrix R for the lattice inclusion $K \subset \Gamma$, and section L for Λ such that $0 \in L$ and $\Lambda = L \dotplus \Gamma^0$, and finally let μ be the associated Hutchinson measure with support $\mathcal{C}$. Then it follows that*

(i) *$\{e_s : s \in K^0\}$ separates points in $\mathcal{C}$, i.e., for $x \neq x'$ in $\mathcal{C}$, $\exists s \in K^0$ s.t. $e_s(x) \neq e_s(x')$.*

(ii) *For each $\ell \in L$, an isometry T_ℓ acting on $L^2(\mu)$ is well defined by*

$$T_\ell e_s = e_{\tau_\ell(s)} \quad \forall s \in K^0 \ . \tag{6}$$

(iii) *As operators on $L^2(\mu)$, the isometries T_ℓ satisfy*

$$T_\ell^* T_{\ell'} = \begin{cases} 0 & \text{if } \ell \neq \ell' \text{ in } L, \\ I & \text{if } \ell = \ell', \end{cases} \qquad \text{and} \qquad \sum_{\ell \in L} T_\ell T_\ell^* = I$$

where I denotes the identity operator on $L^2(\mu)$.

(iv) *The representation of the Cuntz C^*-algebra $\mathcal{O}(L)$ generated by the isometries in* (iii) (*see* [Cu]) *has a canonical factor decomposition associated with the triple $K \subset A \subset \Gamma$ of lattices and the* (*dual*) *fractal measure μ may be reconstructed directly from the associated factor state on $\mathcal{O}(L)$ of the decomposition.* (*Note that the decomposition is orthogonal, and in the category of representations of C^*-algebras*; *see* [BR]).

(v) *The cyclic e_o-representation of $\mathcal{O}(L)$ by the T_ℓ isometries is the GNS representation* (*see* [BR]) *of the factor state ω on $\mathcal{O}(L)$ which is determined by the relations in* (iii), *$\omega(I) = 1$, and $\omega(T_o T_o^*) = 1 = \omega(T_0)$.*

(vi) *The set of all vectors*

$$\{e_o\} \cup \bigcup_{n=1}^{\infty} \{T_{\ell_1} \cdots T_{\ell_n} e_o : \ell_i \in L\} \tag{7}$$

is maximal μ-orthogonal and spans a closed subspace in $L^2(\mu)$ with infinite-dimensional orthogonal complement.

(vii) *The Fourier transform*

$$\widehat{\mu}(t) = \int_{\mathcal{C}} e_t(x) \, d\mu(x)$$

satisfies the functional transformation law

$$\widehat{\mu}(Rt) = B(Rt)\widehat{\mu}(t) \quad \forall t \in \mathbb{R}^d \ ,$$

where

$$B(t) = \frac{1}{N} \sum_{b \in \mathcal{B}} e^{i2\pi b \cdot t}$$

and $\widehat{\mu}(\cdot)$ has an associated infinite product-formula.

Remark. We view the representation (6) as a substitute for an orthogonal harmonic analysis for $L^2(\mu)$, with μ fractal, and note that the relations in (iii) above have the flavor of an orthogonal double-decomposition but *not* an orthogonal expansion in the classical sense of Fourier integrals (or series). Indeed, Strichartz [St2] showed that there is not a direct way of making an exact classical Fourier decomposition for $L^2(\mu)$ when μ is fractal.

4. Orthogonal frequencies in $L^2(\mu)$

Note that in (vi) the vectors from (7) are represented by orthogonal frequencies e_ξ of the form (1) where ξ is in the subset $\mathcal{L}(L) \subset \mathbb{R}^d$ of all affine sums (with n variable):

$$\sum_{k=1}^{n} R^{k-1}\ell_k = \tau_{\ell_1}\tau_{\ell_2}\cdots\tau_{\ell_n}(0) ,$$

$\forall \ell_k \in L$, and the $n = 0$ term corresponding (by definition) to $\xi = 0$.

Theorem 3 (details [JP3]). (i) *$\{e_\xi : \xi \in \mathcal{L}(L)\}$ is maximally orthogonal in $L^2(\mu)$.*

(ii) *None of the functions $e_s(x) = e^{i2\pi s\cdot x}(x \in \mathbb{R}^d)$ for $s \in \mathbb{R}^d\backslash\mathcal{L}(L)$ is in the $L^2(\mu)$-closed linear span of the pure frequencies of $\mathcal{L}(L)$. That is,*

$$\sigma_L(s) := \sum_{\xi\in\mathcal{L}(L)} |\widehat{\mu}(s-\xi)|^2 < 1$$

when s is in $\mathbb{R}^d\backslash\mathcal{L}(L)$.

However, computer-calculations (Mathematica) show that

$$\sigma_L(s) = \|P_{\mathcal{L}(L)}e_s\|^2_{L^2(\mu)}$$

is close to 1 (within third decimal place) when $s = (s_1, \ldots, s_d) \in K^0\backslash\mathcal{L}(L)$ and $s_i > 0$, $1 \leq i \leq d$.

5. Returning to (Ω, Λ)

Our final result shows that the system (Ω, Λ) may be reconstructed from a given Cuntz-representation acting on $L^2(\mu)$.

Theorem 4. *Let μ be a probability measure on $\mathbb{R}^d$ with compact support, and let $K \subset \Gamma$ be a rank d lattice system, with inclusion matrix R. Suppose a subset L s.t. $0 \in L \subset K^0$ induces operators $\{T_\ell\}_{\ell\in L}$ by* (6)*, acting isometrically on $L^2(\mu)$ and satisfying the Cuntz-relations* (iii) *in Theorem* 2*. Then it follows that μ is a fractal measure which is generated by self-similarity from some spectral pair (Ω, Λ) in $\mathbb{R}^d$ satisfying the conditions in Theorem*

1 *for nondegenerate simple factors.*

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