

## THE DINITZ PROBLEM SOLVED FOR RECTANGLES

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ABSTRACT. The Dinitz conjecture states that, for each  $n$  and for every collection of  $n$ -element sets  $S_{ij}$ , an  $n \times n$  partial latin square can be found with the  $(i, j)$ th entry taken from  $S_{ij}$ . The analogous statement for  $(n - 1) \times n$  rectangles is proven here. The proof uses a recent result by Alon and Tarsi and is given in terms of even and odd orientations of graphs.

### I. INTRODUCTION

In 1978, Jeff Dinitz stated a conjecture about partial latin squares; despite the attention it has received from many authors and despite its connection with several other, seemingly unrelated, conjectures, it remains open. The purpose of this paper is to give a proof of the analogous statement for proper latin rectangles.

A *partial latin rectangle* is an  $r \times n$  array of symbols, such that in any row or column, all entries are distinct. If  $r = n$ , the rectangle is referred to as a *partial latin square*. Here is the Dinitz statement:

**Conjecture** (Dinitz). *Suppose that for  $1 \leq i, j \leq n$ ,  $S_{ij}$  is a set of size  $n$ . Then there exists a partial latin square  $L$  such that  $L_{ij} \in S_{ij}$  for all  $i, j$ .*

For a fuller discussion of this conjecture the reader is referred to [CH, ERT, J, K1]. The main objective of this paper is to prove that partial latin rectangles, with the analogous restriction on the entries, always exist. This result is given in the following theorem, which we prove in the next section. The theorem greatly improves a result by Häggkvist [H], which states that partial latin rectangles of size  $r \times n$  with the above property exist for  $r \leq \frac{2}{3}n$ .

**Theorem 1.1.** *Let  $r < n$ , and let  $\mathcal{S} = \{S_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n\}$  be a collection of sets such that  $|S_{ij}| = n$  for all  $i, j$ . Then there exists an  $r \times n$  partial latin rectangle  $L$  with  $L_{ij} \in S_{ij}$  for all  $i, j$ .*

Dinitz's conjecture and Theorem 1.1 are closely related to the list-chromatic index of hypergraphs. The *list-chromatic index*  $\chi'_l(\mathcal{H})$  of a hypergraph  $\mathcal{H}$  is the least number  $t$  such that if each edge  $A$  of  $\mathcal{H}$  is assigned a list  $\mathcal{S}(A)$  of  $t$  "legal" colors, then there is a coloring of the edges of  $\mathcal{H}$  which is proper, i.e., which has the property that no two adjoining edges are assigned the same color and which assigns to each edge  $A$  a color from  $\mathcal{S}(A)$ . Such a coloring is called an  *$\mathcal{S}$ -legal coloring*.

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Now let  $G$  be the rectangular graph of size  $r \times n$ . This is the graph with vertex set  $\{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq n\}$ , where two vertices are connected precisely when they have a coordinate in common. A partial latin square gives a coloring of the vertices of this graph and, hence, also of the edges of the bipartite graph  $K_{r,n}$ , since the line graph of  $K_{r,n}$  is  $G$ . Obviously,  $n$  is a lower bound on the list-chromatic index of  $K_{r,n}$ . From the above discussion it follows that, in terms of the list-chromatic index, Dinitz's conjecture states that  $\chi'_l(K_{n,n}) = n$ . In the same way, Theorem 1.1 translates into the following corollary.

**Corollary 1.1.** *For the bipartite graph  $K_{r,n}$  with  $r < n$ , the list-chromatic index is  $n$ , and for  $K_{n,n}$ , we have that  $\chi'_l(K_{n,n}) \leq n + 1$ .*

For general multigraphs, the best-known bound on the list-chromatic index is by Jeff Kahn. It states that for a hypergraph  $\mathcal{H}$  with bounded edge size and small pairwise degree, and such that any vertex of  $\mathcal{H}$  has degree at most  $D$ ,  $\chi'_l(\mathcal{H}) \leq D + o(D)$ . This bound was conjectured in [K2], and the result is stated and the proof sketched in [K1]. For bipartite graphs this bound implies that  $\chi'_l(K_{r,n}) \leq n + o(n)$ , if  $r \leq n$ . Other bounds are given in [CH, BHa, BHi].

## II. PROOF OF THE MAIN RESULT

We prove Theorem 1.1 using a result proven recently by Alon and Tarsi [AT]. Their main theorem establishes a relation between the number of odd and even orientations of a graph and the existence of  $\mathcal{S}$ -legal colorings of that graph. The statement of the theorem requires some definitions.

Let  $G$  be a graph on an ordered vertex set  $V$ . An *orientation*  $D$  of  $G$  is a directed graph that has the same set of vertices and edges as  $G$ . In a directed graph, an edge  $v \leftarrow w$  such that  $v < w$  is called an *inverted edge*. An orientation  $D$  of  $G$  is called *even* if the number of inverted edges of  $D$  is even and *odd* if the number of inverted edges is odd. For any map  $\delta$  from the set of vertices  $V$  to the nonnegative numbers,  $DE_G(\delta)$  is the number of even orientations of  $G$  such that vertex  $v$  has out-degree (number of out-going edges)  $\delta(v)$ , for each  $v \in V$ .  $DO_G(\delta)$  is the number of odd orientations of  $G$  with the same property. Let  $\mathcal{S} = \{S_v \mid v \in V\}$  be a collection of sets. An  *$\mathcal{S}$ -legal vertex coloring* of  $G$  is a coloring of the vertices of  $G$  which assigns to each vertex  $v$  in  $V$  an element from  $S_v$  and which has the property that no vertices joined by an edge are assigned the same color.

**Theorem 2.1** (Alon-Tarsi). *Let  $G$  be a graph on an ordered vertex set  $V$ . Let  $\mathcal{S} = \{S_v \mid v \in V\}$  be a collection of sets. If there exists a map from the vertex set of  $G$  to the nonnegative integers  $\delta : V \rightarrow \mathbb{Z}^+$  such that  $\delta(v) < |S_v|$  for all  $v \in V$ , and if*

$$DE_G(\delta) \neq DO_G(\delta),$$

*then  $G$  has an  $\mathcal{S}$ -legal vertex coloring.*

We will prove that for rectangular graphs of size  $r \times n$ , with  $r < n$ , we can find a map  $\delta$  such that the conditions of Theorem 2.1 are satisfied, where each  $S_v$  has cardinality  $n$ . We can then invoke this theorem to conclude that there exist  $\mathcal{S}$ -legal vertex colorings of such graphs and, hence, partial latin squares with the desired properties.

Let  $G$  be the rectangular graph of size  $r \times n$ . Let  $D$  be an orientation of  $G$ . Then the *associated matrix*  $L^D$  of  $D$  is the  $r \times n$  matrix with the entry  $L_{ij}^D$  being the

horizontal out-degree of vertex  $(i, j)$ —where the horizontal out-degree of a vertex  $(i, j)$  is the number of edges of type  $(i, j) \rightarrow (i, j')$ . A *latin rectangle* of size  $r \times n$  is an  $r \times n$  matrix with entries taken from  $\{0, 1, \dots, n-1\}$ , with the property that in any row or column no entry is repeated. If  $L$  is an  $r \times n$  latin rectangle, then the *associated orientation*  $D^L$  of  $L$  is the orientation of  $G$  that has  $(i, j) \rightarrow (i, j')$  whenever  $L_{ij} > L_{ij'}$  and  $(i, j) \rightarrow (i', j)$  whenever  $L_{ij} < L_{i'j}$ . Clearly, for all latin rectangles  $L$ , the associated matrix of  $D^L$  is  $L$ .

A *cyclic triangle* is a directed graph on three vertices— $u$ ,  $v$ , and  $w$ —with  $u \rightarrow v \rightarrow w \rightarrow u$ .

**Lemma 2.2.** *Let  $G$  be the complete graph on  $n$  vertices. Then an orientation  $D$  of  $G$  contains a cyclic triangle if and only if there are two vertices of  $D$  that have the same out-degree.*

*Proof.* Let  $D$  be an orientation of  $G$ . Suppose that there are vertices  $u$  and  $v$  of  $D$  that both have out-degree  $a$ . Without loss of generality we can assume that the edge between  $u$  and  $v$  has direction  $u \rightarrow v$ . Since  $G$  is the complete graph, all vertices in  $G$  are contained in  $n-1$  edges. Therefore,  $u$  has  $n-1-a$  incoming edges and  $v$  has  $a$  outgoing edges. But there are only  $n-2$  vertices in  $G$  beside  $u$  and  $v$ ; since  $(n-1-a) + a > n-2$ , there must be at least one vertex  $w$  such that  $w \rightarrow u$  and  $v \rightarrow w$ . Therefore,  $D$  contains a cyclic triangle.

Now suppose that all the out-degrees of the vertices of  $D$  are different.  $G$  can be viewed as the rectangular graph of size  $1 \times n$ . Since all out-degrees  $0, \dots, n-1$  occur exactly once,  $D$  is the associated orientation of a  $1 \times n$  latin rectangle. So a cyclic triangle in  $D$  on vertices  $(1, i)$ ,  $(1, j)$ , and  $(1, k)$  would imply that  $i < j < k < i$ —a contradiction.

The next lemma was inspired by a remark in [AT], where it apparently is tacitly assumed in an argument that establishes an implication from a conjecture about latin squares to Dinitz's conjecture.

**Lemma 2.3.** *Let  $G$  be the rectangular graph with vertex set  $V = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq n\}$ , lexicographically ordered, and let  $\delta : V \rightarrow \mathbb{Z}^+$  be a map from the vertices of  $G$  to the nonnegative integers. Then the number of even orientations of  $G$  that contain a cyclic triangle and have out-degree  $\delta(v)$  at vertex  $v$  for every  $v \in V$  is equal to the number of odd orientations of  $G$  with these properties.*

*Proof.* Let the graph  $G$  and the map  $\delta$  be as in the statement of the lemma. Let  $\mathcal{D}$  be the set of orientations of  $G$  that contain a cyclic triangle and have out-degree  $\delta(v)$  at vertex  $v$  for each  $v$  in  $V$ . Define a map  $\phi : \mathcal{D} \rightarrow \mathcal{D}$  as follows.

For each orientation  $D \in \mathcal{D}$ , let  $(v, w)$  be the lexicographically least pair of vertices such that  $v$  and  $w$  occur in the same row or column and have the same out-degree in the complete subgraph formed by that row or column. Since in a rectangular graph, cyclic triangles can only occur within a row or column, Lemma 2.2 implies that such a pair can always be found. Without loss of generality, we can assume that  $v$  and  $w$  are in the same row, say, row  $k$ , and that  $v \rightarrow w$ . Let the out-degree of  $v$  and, hence, also of  $w$  within the complete subgraph formed by row  $k$  be denoted by  $a$ . Divide the other vertices in row  $k$  into four sets— $V_{oo}$ ,  $V_{oi}$ ,  $V_{io}$ , and  $V_{ii}$ —such that  $V_{oo}$  contains those vertices  $u$  with  $v \rightarrow u \leftarrow w$ ,  $V_{oi}$  the ones with  $v \rightarrow u \rightarrow w$ ,  $V_{io}$  those with  $v \leftarrow u \leftarrow w$ , and  $V_{ii}$  those with  $v \leftarrow u \rightarrow w$ . Note that by counting the number of edges within row  $k$  that go out of  $v$  and  $w$ , we obtain  $a = |V_{oi}| + |V_{oo}| + 1 = |V_{io}| + |V_{oo}|$ , and thus  $|V_{io}| = |V_{oi}| + 1$ .

Now the image of  $D$  under  $\phi$  is the orientation obtained by reversing the direction of the edge  $\langle v, w \rangle$ , all edges between  $v$  and the vertices in  $V_{io} \cup V_{oi}$ , and all edges between  $w$  and the vertices in  $V_{io} \cup V_{oi}$ . The out-degree (in row  $k$ ) of  $v$  in  $\phi(D)$  is  $|V_{oo}| + |V_{io}| = a$ , and that of  $w$  is  $|V_{oo}| + |V_{oi}| + 1 = a$ . The out-degrees in  $\phi(D)$  of the vertices in  $V_{oi} \cup V_{io}$  are the same as in  $D$ , since at each vertex the directions of one out-going and one in-coming edge are reversed. Hence,  $\phi(D) \in \mathcal{D}$ .

In  $\phi(D)$ ,  $(v, w)$  is still the lexicographically least pair of vertices that occur in the same row or column and have the same out-degree in the complete subgraph formed by that row or column. Also,  $V_{oi} \cup V_{io}$  in  $\phi(D)$  is the same as in  $D$ , since reversing the direction of the edge  $\langle v, w \rangle$  switches the roles of  $v$  and  $w$  and, hence, switches  $V_{oi}$  and  $V_{io}$ . It follows that  $\phi$  is an inversion. The number of edges inverted by  $\phi$  is  $2|V_{oi} \cup V_{io}| + 1$ —an odd number—so  $\phi$  maps even orientations to odd ones and vice versa. This shows that  $\phi$  gives a one-to-one correspondence between the odd and even orientations in  $\mathcal{D}$ .

*Remarks.* (1) The “obvious” map of  $\mathcal{D}$  into itself—namely, the one that reverses the direction of the lexicographically first cyclic triangle—is *not*, in general, an involution.

(2) Clearly the crux of the proof above is in showing that for the complete graph on  $m$  vertices, the number of odd orientations with a cyclic triangle is the same as the number of even orientations with a cyclic triangle. Assmus has given a nice proof of this result that proceeds by induction on  $m$ ; that proof—and a discussion of the trouble with the “obvious” proof—is contained in [J].

The circulant  $r \times n$  latin rectangle of order  $n$  is the  $r \times n$  matrix that has  $i + j - 2 \pmod n$  as its  $(i, j)$ th entry.

**Lemma 2.4.** *Let  $r < n$ , and let  $G$  be the rectangular graph of size  $r \times n$  with vertex set  $V = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq n\}$ . Define the map  $\delta : V \rightarrow \mathbb{Z}^+$  as*

$$\delta((i, j)) = \begin{cases} r - 2 + j & \text{for } j \leq n - r + 1, \\ n - 1 & \text{for } n - r + 1 < j \leq n - i + 1, \\ r - 1 & \text{for } j > n - i + 1. \end{cases}$$

*Then the only orientation  $D$  of  $G$  with each vertex  $(i, j)$  of out-degree  $\delta((i, j))$  that does not contain a cyclic triangle is the orientation associated with the circulant latin rectangle.*

*Proof.* It is easy to check that the orientation associated with the circulant  $r \times n$  latin rectangle has out-degree  $\delta((i, j))$  at vertex  $(i, j)$ .

Fix  $n$ . The proof is by induction on  $r$ , where  $r < n$ . For a  $1 \times n$  rectangular graph  $G$ ,  $\delta((1, j)) = j - 1$  for all  $j = 1, \dots, n$ , and hence  $D$  is the orientation associated with the  $1 \times n$  matrix

$$0 \quad 1 \quad \cdots \quad n - 1,$$

which contains no cyclic triangle. Now suppose the lemma is proven for rectangular graphs of size  $(r - 1) \times n$ , and let  $G$  be the rectangular graph of size  $r \times n$ , where  $r < n$ . Let  $D$  be an orientation of  $G$  with out-degrees according to  $\delta$  and without a cyclic triangle. The out-degrees of the row  $r$  of  $D$  are

$$r - 1 \quad r \quad \cdots \quad n - 2 \quad n - 1 \quad r - 1 \quad \cdots \quad r - 1.$$

In the complete subgraph given by row  $r$  of  $D$ , all out-degrees  $0, 1, \dots, n-1$  each have to occur once, since  $D$  contains no cyclic triangle (Lemma 2.2). Since  $r < n$ , there is only one vertex in row  $r$ , namely,  $(r, n-r+1)$ , that has  $\delta$ -value  $n-1$ . Therefore, this vertex must have out-degree  $n-1$  in the subgraph given by row  $r$ , and consequently all vertical edges containing this vertex must be in-coming. The same reasoning can be used to show that all vertices  $v = (r, 2), \dots, v = (r, n-r+1)$  have out-degree  $\delta(v)$  in the subgraph given by row  $r$  and, thus, have only in-coming vertical edges. The remaining vertices all have  $\delta$ -value  $r-1$ .

The out-degrees in the first row of  $D$  are

$$r-1 \quad r \quad \cdots \quad n-1 \quad n-1 \quad \cdots \quad n-1.$$

The lowest out-degree that occurs in this row is  $r-1$ , and since  $r < n$ , it occurs only at vertex  $(1, 1)$ . Since at most  $r-1$  vertices can go out vertically, this vertex must be the one that has out-degree 0 in the complete subgraph given by row 1, and all vertical edges that contain this vertex must be out-going. The lowest out-degree in the second row of  $D$  is also  $r-1$ , and it occurs in the first and the  $n$ th columns. But the vertex in the first column has at most  $r-2$  out-going vertical edges, so the vertex  $(2, n)$  is the one that must have horizontal out-degree 0, and all vertical edges containing this vertex must be out-going. This argument can be pursued further to show that all vertices  $(i, j)$  with  $i+j \equiv 2 \pmod n$  have horizontal out-degree (out-degree in the subgraph given by the row in which they are contained) equal to 0. In particular, vertex  $(r, n-r+2)$  is the one having out-degree 0 in row  $r$ , and all vertical edges containing it are out-going.

Now in the first row the lowest remaining out-degree is  $r$ , and it only occurs at vertex  $(1, 2)$ ; so this vertex must have horizontal out-degree 1, and all vertical edges containing this vertex must be out-going. In the second row the out-degrees  $r-1$  and  $r$  remain in column 1 and 2, respectively. But both vertices have at least one incoming vertical edge, so the only vertex that possibly can have horizontal out-degree 1 is vertex  $(2, 1)$ . Again the reasoning can be followed to show that all vertices  $(i, j)$  where  $i+j \equiv 3 \pmod n$  have horizontal out-degree 1. In particular, vertex  $(r, n-r+3)$  has out-degree 1 in row  $r$ . It has one in-coming vertical edge, from vertex  $(r-1, n-r+3)$ , and all other vertical edges containing it are out-going. We can continue this argument to show that vertex  $(r, j)$  has horizontal out-degree  $j-n+r-2$  for  $j \leq n-r+2 \leq n$ , with out-going edges to vertices  $(1, j), \dots, (n-j+1, j)$  and in-coming edges to the other vertices in its column. The only remaining vertex in row  $r$ — $(r, 1)$ —must therefore have horizontal out-degree  $r-1$ , the only value not yet used, and all vertical vertices containing this vertex are in-coming.

We have proved that the horizontal out-degrees in the complete graph given by row  $r$  are given by

$$r-1 \quad r \quad \cdots \quad n-1 \quad 0 \quad 1 \quad \cdots \quad r-2,$$

and thus row  $r$  of the matrix associated with  $D$  is equal to row  $r$  of the  $r \times n$  circulant latin rectangle. From the argument above we can conclude that the rectangular subgraph of size  $(r-1) \times n$  given by the first  $r-1$  rows of  $G$  must have out-degrees according to  $\delta'$ ; here  $\delta'((i, j)) = \delta((i, j)) - 1$  for  $1 \leq i \leq r-1$  and  $j \leq n-r+1$  or  $j > n-i+1$ , and  $\delta'((i, j)) = \delta((i, j))$  for other values of  $i, j$ . It can be checked that

these values agree with the definition of  $\delta$  when  $r$  is substituted by  $r - 1$ . Therefore, we can use the induction hypothesis to show that the first  $r - 1$  rows of the matrix associated with  $D$  are also equal to the corresponding rows of the circulant latin rectangle.

We are now able to prove our main result.

*Proof of Theorem 1.1.* Let  $G$  be the rectangular graph of size  $r \times n$ , on vertex set  $V$ . Order the vertices of  $G$  according to the following rule:  $(i, j) < (i', j')$  precisely when  $i < i'$  or  $i = i'$  and  $j > j'$ . Let  $\mathcal{S} = \{S_v \mid v \in V\}$  be a collection of sets such that  $|S_v| = n$  for each  $v \in V$ . Let the map  $\delta : V \rightarrow \mathbb{Z}^+$  be as in the statement of Lemma 2.4. Note that  $\delta(v) < n$  for all  $v$ . By Lemma 2.3, the number of even orientations with out-degrees corresponding to  $\delta$  that contain a cyclic triangle is equal to the number of odd orientations with these properties, and by Lemma 2.4 there is precisely one orientation of  $G$  that does not contain a cyclic triangle. So  $DE_G(\delta) - DO_G(\delta) = 1$  or  $-1$ . Hence, by Theorem 2.1, there exists an  $\mathcal{S}$ -legal coloring of  $G$ . The corresponding  $r \times n$  matrix  $L$  that has the color assigned to vertex  $(i, j)$  of  $G$  as its  $L_{ij}$  entry forms a partial latin rectangle with the property that  $L_{ij} \in S_{ij}$  for all  $i, j$ .

Using Theorem 1.1, we can also prove a weaker version of Dinitz's conjecture. The following theorem justifies the second half of Corollary 1.1.

**Theorem 2.4.** *Suppose that for  $1 \leq i, j \leq n$ ,  $S_{ij}$  is a set of size  $n + 1$ . Then there exists a partial latin square  $L$  such that  $L_{ij} \in S_{ij}$  for all  $1 \leq i, j \leq n$ .*

*Proof.* Let  $S_{ij}$  be a set of size  $n + 1$ , for  $1 \leq i, j \leq n$ . Set  $S_{i,n+1} = S_{i,n}$  for  $1 \leq i \leq n$ . Now, by Theorem 1.1, there exists an  $n \times (n + 1)$  latin rectangle  $L$  such that  $L_{ij} \in S_{ij}$  for all  $1 \leq i \leq n$ ,  $1 \leq j \leq n + 1$ . If we delete the last column of  $L$ , we obtain a partial latin square with the desired property.

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