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BOOK REVIEW

Groups, generators, syzygies, and orbits in invariant theory, by V. L. Popov (translated from the Russian by A. Martsinkovsky). Translations of Mathematical Monographs, vol. 100, American Mathematical Society, Providence, RI, 1992, vi + 245 pp., $181.00. ISBN 0-8218-4557-8

1. The book

The book in question is a translation of some of V. Popov's most important and significant early papers, all written prior to 1984. They were the basis for awarding Popov the Doctor of Science degree, a degree beyond the usual Ph.D. equivalent to the German Habilitation. When Popov learned of the interest of the American Mathematical Society in translating his work, he generously added an appendix to bring matters up to date.

There are a large number of topics in invariant theory that are touched upon. I have decided to devote most of my description to those that are connected to computing invariants for semisimple groups, which comprise about half of the book. I will briefly mention the other topics but will hardly do them justice.

2. Hilbert's theorem

Our base field is the field of complex numbers $\mathbb{C}$. The only topology we consider is the Zariski topology. We consider affine algebraic varieties in the naive sense, as closed irreducible algebraic subsets X of some $\mathbb{C}^n$ with their natural algebra of polynomial functions $\mathbb{C}[X]$.

Let G be a connected semisimple algebraic group. For example, G could be $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SO}_n(\mathbb{C})$. By a representation of G we mean a homomorphism of algebraic groups $G \to \mathrm{GL}(V)$ for some finite-dimensional vector space V. We refer to V with the induced action of G as a *G-module.* If $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SO}_n(\mathbb{C})$, then we have the canonical action of G on $W := \mathbb{C}^n$. We can make more complicated representations by taking tensor powers, exterior powers, etc., of W and direct sums of these.

Let V be a G-module. The *orbit* of $v \in V$, denoted Gv, is $\{gv : g \in G\}$, and the *stabilizer* or *isotropy group of* v is $\{g \in G : gv = v\}$. Thus $Gv \simeq G/G_v$.

There are several classical problems concerning the G-action on V:

Problem 1. What are the orbits of the G-action on V?

Problem 2. What reasonable structure can one put on the set of G-orbits?

These problems are too difficult to be reasonably solved. The set of G-orbits is simply too "wild". However, if one restricts oneself to the set of *closed orbits*, then one gets, at least theoretically, a good solution to Problem 2, as we see shortly.

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Theorem 1 (Hilbert, essentially). *Let G and V be as above. Then the algebra $\mathbb{C}[V]^G$ is finitely generated.*

Let $V/\!\!/G$ denote the complex affine variety corresponding to the algebra $\mathbb{C}[V]^G$, and let $\pi_V : V \to V/\!\!/G$ denote the morphism dual to the inclusion $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$. To be brutally specific, let $p_1, \dots, p_d$ be generators of $\mathbb{C}[V]^G$, and let $p = (p_1, \dots, p_d) : V \to \mathbb{C}^d$. Let J denote the kernel of $p^* : \mathbb{C}[Y_1, \dots, Y_d] \to \mathbb{C}[V]^G$, where $p^*(Y_i) = p_i$. (J is the "ideal of relations" of the p_i.) Let $Z \subseteq \mathbb{C}^d$ be the zero set of J. Then $p : V \to Z$ and, by construction, there is an isomorphism $\varphi : V/\!\!/G \xrightarrow{\sim} Z$ and a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi_V} & V/\!\!/G \\ & & \sim \big\downarrow \varphi \\ V & \xrightarrow{p} & Z \end{array}$$

Theorem 2. (1) *π_V and p are surjective.*

(2) *π_V and p separate disjoint closed G-stable subsets of V.*

(3) *Every orbit contains a unique closed orbit in its closure, and π_V (resp. p) sets up a bijection between the closed orbits of V and the points of $V/\!\!/G$ (resp. Z).*

From now on we concentrate on trying to say something about $\mathbb{C}[V]^G$ and $V/\!\!/G$, partially addressing Problem 2. If one wants to study Problem 1, one has to consider the fibers of the morphism π_V, or more generally, one has to study all affine G-varieties which contain a single closed orbit.

Example. Let $G = \mathrm{SL}_2(\mathbb{C})$ and $V = 4\mathbb{C}^2$, the direct sum of four copies of the standard 2-dimensional module. We denote points $v \in V$ as 4-tuples $(v_1, \dots, v_4) \in 4\mathbb{C}^2$. Then the functions $p_{ij}(v) := \det(v_i, v_j)$ are in $\mathbb{C}[V]^G$, $1 \le i < j \le 4$. It is known (see the exercise below) that

(1) the p_{ij} generate $\mathbb{C}[V]^G$; and

(2) the ideal of relations of the p_{ij} is principal, generated by

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

Note that the function in (2) is skew symmetric in the four copies of $\mathbb{C}^2$; hence, it must be 0 for dimensional reasons.

In the notation given after Theorem 1, we have

$$p := (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) : V \to \mathbb{C}^6,$$

the ideal $J \subseteq \mathbb{C}[Y_{12}, \dots, Y_{34}]$ is generated by $Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}$, and $p : V \to \mathbb{C}^6$ has image $Z = \{\text{zeros of } J\}$.

It is not too hard to show that an orbit $G(v_1, v_2) \in 2\mathbb{C}^2$ is closed iff $(v_1, v_2) = 0$ or v_1 and v_2 are linearly independent (in which case $G(v_1, v_2) = \{(w_1, w_2) : \det(w_1, w_2) = \det(v_1, v_2)\}$). It follows that a point $0 \ne v = (v_1, \dots, v_4) \in V$ lies on a closed G-orbit iff $\dim \operatorname{span}\{v_1, \dots, v_4\} = 2$ iff $p_{ij}(v) \ne 0$ for some i and j. Thus the fibers $p^{-1}(z)$, $z \ne 0$, are all closed G-orbits. The *null cone* $\mathcal{N}_V := p^{-1}(0)$ consists of those v such that $\dim \operatorname{span}\{v_1, \dots, v_4\} < 2$. All the (infinitely many) orbits in $\mathcal{N}_V$ have $\{0\}$ in their closure, and $\{0\}$ is the only closed orbit in $\mathcal{N}_V$.

3. Constructive invariant theory

Hilbert already considered the following problem. It was not properly addressed before Popov's work:

Problem 3. Given the G-module V, find a *computable* number m_V such that one can find generators $p_1, \ldots, p_d$ of $\mathbb{C}[V]^G$ of degrees at most m_V.

Once one has found the number m_V, then the p_i lie in the finite-dimensional space $\bigoplus_{1\le i\le m_V} \mathbb{C}[V]_i^G$, where the subscript i indicates polynomials homogeneous of degree i. The problem of finding the generators of $\mathbb{C}[V]^G$ is reduced to a finite-dimensional problem. (Of course, the dimension is usually greater than the number of electrons in the universe or any other of your favorite large numbers.)

For a long time Theorem 1 was the main important result known about $\mathbb{C}[V]^G$, but this changed in 1974 with the Hochster-Roberts [HR] theorem. It is an essential ingredient in Popov's construction of m_V. We give the theorem in its most down-to-earth form:

Theorem 3. *There are homogeneous polynomials $f_1, \ldots, f_r$ such that $\mathbb{C}[V]^G$ is a finite free $\mathbb{C}[F] := \mathbb{C}[f_1, \ldots, f_r]$-module.*

The integer r is the Krull dimension of $\mathbb{C}[V]^G$, which equals $\dim V - \max\{\dim Gv : v \in V\}$. In general, $r = \dim V - \dim G$.

Let $h_0, \ldots, h_s$ be homogeneous such that $\mathbb{C}[V]^G \simeq \mathbb{C}[F]h_0 \oplus \cdots \oplus \mathbb{C}[F]h_s$. Then the Poincaré series $P(\mathbb{C}[V]^G, t) := \sum_{i=0}^{\infty} \dim_{\mathbb{C}} \mathbb{C}[V]_i^G t^i$ has the form

$$P(\mathbb{C}[V]^G, t) = \sum_{j=0}^{s} \frac{t^{e_j}}{(1-t^{d_1})\cdots(1-t^{d_r})} = \sum_{k=0}^{e} \frac{a_k t^k}{(1-t^{d_1})\cdots(1-t^{d_r})}$$

where $\deg f_i = d_i$, $\deg h_j = e_j$, $e = \max\{e_j\}$, and $a_k = \#\{j : e_j = k\}$. Since the f_i and h_j generate $\mathbb{C}[V]^G$, we obtain $m_V \le \max\{d_i, e\}$, and it suffices to bound the d_i and e.

Remark. Since G is connected semisimple, $\mathbb{C}[V]^G$ is Gorenstein [Mu]. Given that $\mathbb{C}[V]^G$ is Cohen-Macaulay, this is equivalent to the fact that $a_i = a_{e-i}$, $0 \le i \le e$ [St1].

Let q denote minus the degree of the rational function $P(\mathbb{C}[V]^G, t)$. Then $q = (\sum_i d_i) - e$, and estimating q is equivalent to estimating e, assuming one knows the d_i. Kempf [Ke] showed that $q \ge 0$. Using results of Stanley (see [St2]), Popov showed that $q = \dim V$ for "almost all" representations of G, and he conjectured that $q \le \dim V$ in general. Knop [Kn], expanding upon ideas of Panyushev [Pa], established:

Theorem 4. (1) $\dim \mathbb{C}[V]^G \le q \le \dim V$.

(2) $q = \dim V$ *if* $\operatorname{codim}_V(V \setminus V') \ge 2$, *where V' denotes the union of the orbits of V with finite stabilizer.*

Theorem 4 implies that $m_V \le \sum_i d_i$, so one needs a computable bound for the degrees of the f_i. Popov finds an integer n_V, given by a formula involving the weights of V, the dimension of V, and the dimension and rank of G, with the following property: For every $\xi \in V/\!\!/G \setminus \pi_V(0)$ there is a nonconstant homogeneous $f_\xi \in \mathbb{C}[V]^G$ of degree at most n_V such that $f(\xi) \ne 0$. If it were true that for every ξ we could choose f_ξ to be homogeneous of degree n_V, then by the Noether normalization lemma we could find $f_1, \dots, f_r$ as in Theorem 3, all of degree n_V, which would give $m_V \leq r n_V$. Unfortunately, one cannot guarantee this in general, and one has to replace $r n_V$ by r times the least common multiple of $\{2, 3, \dots, n_V\}$. The number n_V is huge itself; it grows faster than $(\dim V)^{\dim G}$ as $\dim V \to \infty$.

Exercise. Consider our example, where $G = \mathrm{SL}_2(\mathbb{C})$ and $V = 4\mathbb{C}^2$. Use the Remark and Theorem 4 to compute the Poincaré series $P(\mathbb{C}[V]^G, t)$. Use it to show that the p_{ij} generate $\mathbb{C}[V]^G$ and that their ideal of relations is as claimed.

4. Complicated algebras of invariants

Even though the algebra of invariants is effectively computable, in practice, the computations are beyond our grasp. The numbers are simply too large. Invariant theorists have found several cases where the algebra of invariants has an easy description, and all other cases seem hopelessly complicated. One can measure the complicatedness of $\mathbb{C}[V]^G$ in several ways, one being its *embedding codimension.*

Let $p_1, \dots, p_d$ be minimal generators of $\mathbb{C}[V]^G$. Then $V/\!\!/G$ embeds into $\mathbb{C}^d$ (and not into $\mathbb{C}^{d-1}$). One defines the *embedding codimension* $\operatorname{ec}\mathbb{C}[V]^G$ of $\mathbb{C}[V]^G$ (and embedding codimension of $V/\!\!/G$) to be $d - \dim \mathbb{C}[V]^G$. This is the same as the homological dimension of $\mathbb{C}[V]^G$ as a quotient ring of $\mathbb{C}[X_1, \dots, X_d]$, where $X_i \mapsto p_i$, $i = 1, \dots, d$. If $\operatorname{ec}\mathbb{C}[V]^G = 0$, then $\mathbb{C}[V]^G \simeq \mathbb{C}[X_1, \dots, X_d]$, and we say that V is *coregular.* Popov showed:

Theorem 5. *Consider G-modules V such that $V^G = (0)$, and fix an integer $t \geq 0$. Then, up to isomorphism, there are only finitely many V such that $\operatorname{ec}\mathbb{C}[V]^G \leq t$. In particular, up to isomorphism, there are only finitely many V such that V is coregular.*

In the case $t = 0$, one can easily establish the result using Theorem 4: Let $p_1, \dots, p_d$ be a minimal homogeneous set of generators. Then $d_i := \deg p_i \geq 2$ for each i (because $V^G = (0)$). Since $d \geq \dim V - \dim G$ and V is coregular (i.e., $e = 0$), an easy manipulation gives that $\dim V \leq 2 \dim G$. Since there are only finitely many isomorphism classes of representations of dimension at most k for any k, one obtains the finiteness result.

In the case $t > 0$, one needs a different argument. In fact, one cannot even go on without mentioning (a consequence of) Luna's slice theorem: Let $\pi_V \colon V \to V/\!\!/G$ be as in §2. One can ask the following question: What does $V/\!\!/G$ look like near $\pi_V(v)$, Gv closed? By a theorem of Matsushima, G_v is reductive; i.e., every G_v-module can be written as a sum of irreducible G_v-modules. Now V is a G_v-module, and the tangent space $T_v(Gv)$ to the orbit Gv at v can be identified with a G_v-stable subspace of V by translation to the origin. By reductivity there is a G_v-stable complement N_v to $T_v(G_v)$, and the representation $G_v \to \mathrm{GL}(N_v)$ is called the *slice representation at v.* Luna's slice theorem implies that a neighborhood of $\pi_V(v)$ in $V/\!\!/G$ "looks like" a neighborhood of the origin in $N_v/\!\!/G_v$. I will not make "looks like" precise, except to say that if one changes to the usual topology, one obtains an analytic equivalence.

Since $V/\!\!/G$ is a cone (due to the natural $\mathbb{C}^*$-action), the singularity of $V/\!\!/G$ at the vertex $\pi_V(0)$ is worse than the singularity at $\pi_V(v)$. Thus V is at least as "bad" as N_v. In particular, the embedding codimension of $\mathbb{C}[V]^G$ is at least that of

$\mathbb{C}[N_v]^{G_v}$. Popov's proof that most representations have high embedding codimension consists in systematically finding "bad" slice representations. Kac, Popov, and Vinberg pioneered this method in their work classifying the irreducible coregular representations of the simple groups. Popov's argument is a vast elaboration on this theme.

Since Popov's work appeared, there has also been work of Gordeev [Go] on the same problem. Gordeev uses the idea of the *class* $\mathrm{cl}(V)$ of a G-module V, where

$$\mathrm{cl}(V) = \min_{g \in G \setminus \{e\}} \{\operatorname{codim} V^g\}.$$

The larger $\mathrm{cl}(V)$ is, the more "general" the action is, and the more complicated is $\mathbb{C}[V]^G$. Here is a simplified and weakened form of what Gordeev shows:

Theorem 6. *Consider G-modules V as in Theorem 5, and fix $t > 0$. Let l denote the rank of G. Then:*

1. $\operatorname{ec} \mathbb{C}[V]^G \geq (\sqrt{\dim V}/24|Z_G|l^2) - 2\dim G - 1$, *where $|Z_G|$ denotes the order of the center Z_G of G.*
2. $\operatorname{ec} \mathbb{C}[V]^G \geq ((\mathrm{cl}\, G)/l^2) - 2\dim G - 1$.

5. Other topics

The foregoing discussion covers in a cursory manner the topics in Chapters 2, 3, and 4 in Popov's book, which is about half of it. The other portions of the book consider and solve (in general or in special cases) the following problems:

Problem 4. For which linear algebraic groups G is it true that $\mathbb{C}[X]^G$ is always finitely generated for X an affine G-variety? (X an affine G-variety means that G acts morphically on X, equivalently, X is a G-stable subvariety of a G-module).

Problem 5. For which V is it true that $\mathbb{C}[V]$ is a free module over $\mathbb{C}[V]^G$?

It turns out that this is equivalent to the following two conditions:

(a) V is coregular.
(b) The morphism $\pi_V : V \to V/\!\!/G$ is equidimensional; equivalently, $\operatorname{codim} \mathcal{N}_V = \dim V/\!\!/G$.

Popov also considered various classification problems for low-dimensional varieties with large automorphism groups. In particular, he solved

Problem 6. Consider normal affine SL_2-varieties which have a dense orbit. Up to isomorphism, how many are there?

Luna and Vust developed techniques to handle the nonaffine case and to handle the case of general G. There has been an explosion of activity in this area over the last ten years. Popov's work was seminal along with earlier work on toroidal embeddings.

6. The book revisited

Popov is a leader in invariant theory, and the articles in this book were important to that field's development. The appendix and introduction are excellent, and I highly recommend them as a starting point for anyone wanting to know more about the subjects touched upon in this review. The list of references is also quite good. What many people will find difficult to swallow, however, is the price of the book—\$181 list, \$109 for AMS members.

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Gerald W. Schwarz
Brandeis University
E-mail address: schwarz@math.brandeis.edu