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PSEUDO-PERIODIC HOMEOMORPHISMS AND DEGENERATION OF RIEMANN SURFACES

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Abstract. We will announce two theorems. The first theorem will classify all topological types of degenerate fibers appearing in one-parameter families of Riemann surfaces, in terms of "pseudoperiodic" surface homeomorphisms. The second theorem will give a complete set of conjugacy invariants for the mapping classes of such homeomorphisms. This latter result implies that Nielsen's set of invariants [*Surface transformation classes of algebraically finite type*, Collected Papers 2, Birkhäuser (1986)] is not complete.

Let $\{F_\xi\}$ be a family of Riemann surfaces parametrized by complex numbers ξ. As ξ approaches a special value, say, 0, F_ξ changes its "shape" and finally gets singularities becoming a singular surface F_0. This degeneration phenomenon has long been studied. Here we study it from the topological point of view. We will show that the topological types of the degenerate fibers can be completely classified in terms of certain surface mapping classes introduced by Nielsen [Ni2] some fifty years ago. We will also give a complete set of conjugacy invariants for such mapping classes.

Throughout this paper all manifolds will be oriented, and all homeomorphisms between them will be orientation-preserving. Σ_g will denote a closed surface of genus g. Details will appear in [MM2].

1. Pseudoperiodic homeomorphisms

A homeomorphism $f : \Sigma_g \to \Sigma_g$ and its mapping class $[f\,]$ are called in this paper *pseudoperiodic* if $[f\,]$ is either of finite order or reducible and in the latter case all component mapping classes are of finite order. (Cf. [Th, G].) It was Nielsen [Ni2] who first studied these mapping classes under the name of *surface*

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transformation classes of algebraically finite type. Let us recall some conjugacy invariants introduced by Nielsen. (See [Ni2, G].)

Suppose f is reduced by a system of simple closed curves $C = C_1 \cup C_2 \cup \cdots \cup C_r$. C is called *admissible* if each connected component of $\Sigma_g - C$ has negative Euler characteristic. If $g \geq 2$, such a system always exists. With each curve C_j of an admissible system C is associated a rational number $s(C_j)$ called the *screw number*. This measures the amount of Dehn twist performed by f^α about C_j, where $\alpha = \alpha(C_j)$ is the smallest positive integer such that $f^\alpha(\overrightarrow{C_j}) = \overrightarrow{C_j}$. An admissible system C is *precise* if $s(C_j) \neq 0$ for each C_j. A precise system always exists and is unique up to isotopy.

A curve C_j is *amphidrome* if α is even and $f^{\alpha/2}(\overrightarrow{C_j}) = -\overrightarrow{C_j}$.

We say that a pseudoperiodic homeomorphism f is *of negative twist* if either $[f\,]$ is of finite order, or, when $[f\,]$ is reducible, $s(C_j) < 0$ for each curve C_j in a precise system C.

2. Degenerating family

By a *degenerating family* (*of Riemann surfaces*) *of genus* g we mean a triple (M, D, φ) consisting of a noncompact complex surface M; an open unit disk $D = \{\xi \in \mathbb{C} \mid |\xi| < 1\}$; and a surjective, proper, and holomorphic map $\varphi : M \to D$. All fibers $F_\xi = \varphi^{-1}(\xi)$ are assumed to be connected, and outside the origin $\varphi|\varphi^{-1}(D^*) : \varphi^{-1}(D^*) \to D^*$ is assumed to be a smooth fiber bundle with fiber Σ_g , where $D^* = D - \{0\}$. The family is *minimal* if it is free from (-1)-curves. Two families, $(M_i, D_i, \varphi_i), i = 1, 2$, are *topologically equivalent* ($\overset{\text{TOP}}{\sim}$) if there exist homeomorphisms $H : M_1 \to M_2$ and $h : D_1 \to D_2$ satisfying $h(0) = 0$ and $h\varphi_1 = \varphi_2 H$. We are interested in the following set:

$$\mathcal{S}_g = \{\text{minimal degenerating families of genus } g\}/\overset{\text{TOP}}{\sim}.$$

Given a degenerating family of genus g, the *monodromy homeomorphism* $f : \Sigma_g \to \Sigma_g$ around the central fiber F_0 is determined as usual (up to isotopy and conjugation). By the results of Imayoshi [I], Shiga and Tanigawa [ST], and Earle and Sipe [ES], f is a pseudoperiodic homeomorphism of negative twist. (There is an alternative topological proof, [MM2].) Let $\mathcal{M}_g$ be the mapping class group of Σ_g and $\widehat{\mathcal{M}}_g$ the set of conjugacy classes in $\mathcal{M}_g$. Let $\mathcal{P}_g^-$ denote the subset of $\widehat{\mathcal{M}}_g$ represented by pseudoperiodic mapping classes of negative twist. Then we have a well-defined map

$$\text{monodromy } \rho : \mathcal{S}_g \to \mathcal{P}_g^-.$$

Theorem 1. *For $g \geq 2$, $\rho : \mathcal{S}_g \to \mathcal{P}_g^-$ is bijective.*

The corresponding map for $g = 1$ is surjective, and the "kernel" consists of multiple fibers. (Cf. [K].) Using Theorem 1 and the construction in §3, one can topologically recover Namikawa and Ueno's classification of singular fibers of genus 2 [NU]. Since $\mathcal{M}_g \cong \mathrm{Aut}(\pi_1\Sigma_g)/\mathrm{Inn}(\pi_1\Sigma_g)$, we have

Corollary 1.1. *If $g \geq 2$, the action of the monodromy on $\pi_1\Sigma_g$ determines the topological equivalence class of (M, D, φ). In particular, if the action is trivial, F_0 is nonsingular.*

Note that the action on $H_1(\Sigma_g;\mathbb{Z})$ is not sufficient [NU]. For an explicit algebraic calculation of nonabelian monodromy see [O].

Corollary 1.2. *Given a pseudoperiodic homeomorphism of negative twist $f : \Sigma_g \to \Sigma_g$, there exists a degenerating family (M, D, φ) whose monodromy homeomorphism coincides with f up to isotopy and conjugation.*

A closely related existence theorem has been independently announced by Earle and Sipe [ES, §7]. See [MM1] for a short abstract of our result, where we adopted a sign convention opposite to the one here.

3. Generalized quotient

The idea in proving Theorem 1 is to construct the inverse map of $\rho : \mathcal{S}_g \to \mathcal{P}_g^-$. A *Riemann surface with nodes* S was introduced by Bers [B]. We will call the underlying topological space of S a *chorizo space* (chorizo = Spanish sausage), which we allow to have boundaries and not to be connected. A chorizo space below will be *numerical* in the sense that to each irreducible component is attached a positive integer called the *multiplicity*.

For a pseudoperiodic homeomorphism of negative twist $f : \Sigma_g \to \Sigma_g$ we can construct a numerical chorizo space called the *generalized quotient* S_f of f as follows:

Decompose Σ_g as $\Sigma_g = A \cup B$, where A is the union of annular neighborhoods of the curves in the precise system C such that $f(A) = A$. We assume that $f|B : B \to B$ is periodic. The quotient space $B/(f|B)$ is an orbifold. Let p be a cone point, (m, λ, σ) the *valency* of p [Ni1]; that is, if $x \in B$ is a point over p, m is the smallest positive integer such that $f^m(x) = x$, f^m is the rotation around x through the angle $2\pi\delta/\lambda$ $(0 < \delta < \lambda, \gcd(\lambda, \delta) = 1)$ and σ is the integer determined by $\delta\sigma \equiv 1 \pmod{\lambda}$, $0 < \sigma < \lambda$.

By the Euclidean algorithm we obtain a sequence of integers $n_0 > n_1 > \cdots > n_l = 1$ such that $n_0 = \lambda, n_1 = \lambda - \sigma, n_{i-1} + n_{i+1} \equiv 0 \pmod{n_i}, i = 1, \ldots, l-1$. Set $m_i = mn_i$ $(i = 0, 1, \ldots, l)$.

Let $\mathrm{Ch}(B)$ be the chorizo space constructed from $B/(f|B)$ by replacing a neighborhood of each cone point with the numerical chorizo space shown in Figure 1, which consists of a disk and l spheres.

Let $A_j(\subset A)$ be an annular neighborhood of C_j. The boundary curves S_1 and S_2 of A_j have their *valencies* $(m^{(1)}, \lambda^{(1)}, \sigma^{(1)})$ and $(m^{(2)}, \lambda^{(2)}, \sigma^{(2)})$, when regarded as boundary curves of the periodic part B [Ni1].

Suppose C_j is *not* amphidrome. Then $m^{(1)} = m^{(2)} = \alpha(C_j)$. Let m be this common value.

Lemma. *There exists uniquely a sequence of positive integers $n_0, n_1, \ldots, n_l$ $(l \geq 1)$ satisfying the following conditions*:

(i) $n_0 = \lambda^{(1)}, n_l = \lambda^{(2)}$;
(ii) $n_1 \equiv \sigma^{(1)} (\mathrm{mod}\ \lambda^{(1)}), n_{l-1} \equiv \sigma^{(2)} \pmod{\lambda^{(2)}}$;
(iii) $n_{i-1} + n_{i+1} \equiv 0 \pmod{n_i}, i = 1, 2, \ldots, l-1$;
(iv) $(n_{i-1} + n_{i+1})/n_i \geq 2, i = 1, 2, \ldots, l-1$; *and*
(v) $\sum_{i=0}^{l-1} 1/n_i n_{i+1} = |s(C_j)|$.

Let $\mathrm{Ch}(A_j)$ be the chorizo space shown in Figure 2, which consists of two disks and $l-1$ spheres, m_i being defined to be mn_i.

Figure 1

Figure 2

We consider the spaces $\mathrm{Ch}(f^i A_j)$ $(i = 0, 1, \ldots, m-1)$ identical : $\mathrm{Ch}(A_j) = \mathrm{Ch}(fA_j) = \cdots = \mathrm{Ch}(f^{m-1}A_j)$.

Finally, suppose A_j is amphidrome. Then S_1 and S_2 have the same valency $(2m, \lambda, \sigma)$, where $2m = \alpha(C_j)$. Let $n_0, n_1, \ldots, n_l$ be a sequence of integers satisfying $n_0 \geq n_1 \geq \cdots \geq n_l = 1$, $n_0 = \lambda$,$n_1 = \sigma$, $n_{i-1} + n_{i+1} \equiv 0 \pmod{n_i}$, and $\sum_{i=0}^{l-1} 1/n_i n_{i+1} = (1/2)|s(C_j)|$.

Let $\mathrm{Ch}(A_j)$ be the chorizo space shown in Figure 3, which consists of a disk and $l + 2$ spheres. Again we consider the spaces $\mathrm{Ch}(f^i A_j)$ $(i = 0, 1, \ldots, (m/2) - 1)$ identical.

Now the generalized quotient S_f is defined to be the union of $\mathrm{Ch}(B)$ and $\mathrm{Ch}(A_j)$'s , A_j running over all the annuli in A. A natural projection $\pi : \Sigma_g \to S_g$ can be defined.

Let C_π be the mapping cylinder of π. We construct an "open book" $\overline{M}$ with a "page" C_π. (See [Ta, W].) Then $M = \mathrm{int}(\overline{M})$ has a complex structure, and we obtain a degenerating family (M, D, φ) whose monodromy coincides with f. Blow down (-1)-curves, if any, in M. All of the process is topologically canonical, and we get the inverse map $\sigma : \mathcal{P}_g^- \to \mathcal{S}_g$ of $\rho : \mathcal{S}_g \to \mathcal{P}_g^-$, proving its bijectivity.

4. Conjugacy invariants

We define the *partition graph* X_f associated with a pseudoperiodic homeomorphism of negative twist $f : \Sigma_g \to \Sigma_g$ as follows: Let C be a precise system. The vertices (resp. the edges) of X_f are in one-to-one correspondence to the connected components b of $\Sigma_g - C$ (resp. the curves $\{C_i\}$ in C). An edge $e(C_i)$ joins vertices $v(b)$ and $v(b')$ if and only if C_i is in the adherence of b and also of b'. The *refined partition graph* $\overline{X}_f$ is obtained from X_f by subdividing those edges $e(C_i)$ that correspond to amphidrome curves by their middle points.

A periodic map $\psi_f : \overline{X}_f \to \overline{X}_f$ is induced from f. The quotient graph $Y_f = \overline{X}_f / \psi_f$ is a weighted graph in the sense that each vertex (and each edge) carries a positive integer called the *weight*, which is the number of the vertices (resp. the edges) of $\overline{X}_f$ over the vertex (resp. the edge) of Y_f.

The conjugacy class of the periodic action $\psi_f : \overline{X}_f \to \overline{X}_f$ can be interpreted as a cohomology class c_f in a suitably defined *weighted cohomology group* $H^1_W(Y_f)$. The weighted graph Y_f also serves as the *decomposition diagram* of S_f, and there is a natural collapsing map $\eta_f : S_f \to Y_f$.

FIGURE 3

FIGURE 4

Theorem 2. *The triple (S_f, Y_f, c_f) determines the conjugacy class of the mapping class $[f\,]$.*

Nielsen's set of invariants introduced in [Ni2] has exactly the same amount of information as (S_f, Y_f) but lacks c_f. Thus his assertion [Ni2, §15; G, Theorem 13.4] that his invariants are complete is incorrect.

Here is an example. Let Σ_6 be the surface shown in Figure 4, which has a system of curves $C = \{C_1, \dots, C_5\}$. Let $f_k : \Sigma_6 \to \Sigma_6$ $(k = 1, 2)$ be a homeomorphism such that $f_k^5|(\Sigma_6 - C) \simeq \mathrm{id}$, $f_k(b_i) = b_{i+k}$,$f_k(C_j) = C_{j+k}$, and $s(C_j) = -1$. (Indices are taken modulo 5.) Although Nielsen's invariants are the same for $[f_1]$ and $[f_2]$, these mapping classes are not conjugate, because the actions on the partition graphs are not conjugate.

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