

BOOK REVIEW

An extension of Casson's invariant, by Kevin Walker. Princeton University Press, Princeton, NJ, 1992, v+131 pp., \$16.50. ISBN 0-691-02532-0

An *integral* (resp. *rational*) *homology 3-sphere* is a closed 3-manifold whose first integral (resp. rational) homology group vanishes. (We henceforth abbreviate these terms as $\mathbb{Z}HS$ and $\mathbb{Q}HS$ respectively.) These represent an important and slippery class of 3-manifolds—including, for example, those 3-manifolds for which Thurston's tantalizing Geometrization Conjecture has yet defied proof. Furthermore, such 3-manifolds are as plentiful as knots: Given any knot $K \subset S^3$, the various *Dehn fillings* obtained by attaching solid tori to the complement $S^3 - K$ are all homology spheres. Before 1985 the principal invariant of a $\mathbb{Z}HS \Sigma^3$ was the $\mathbb{Z}/2$ -valued *Rohlin invariant* $\mu(\Sigma^3)$, derived from the signature of a bounding 4-manifold. At that time it seemed plausible that the Rohlin invariant could detect a possible counterexample to the Poincaré conjecture.

In 1985 Andrew Casson found a powerful \mathbb{Z} -valued invariant $\lambda(\Sigma^3)$ of a $\mathbb{Z}HS \Sigma^3$, which laid these hopes to rest. From its existence and basic properties follow some remarkable results in geometric topology, such as: *there exist topological 4-manifolds which are not homeomorphic to simplicial complexes*. A closely related result concerns the Poincaré conjecture: Casson's invariant is defined in terms of the fundamental group and reduces modulo two to Rohlin's invariant. In particular, Rohlin's invariant cannot detect a counterexample to the Poincaré conjecture.

For an exposition of Casson's original results, presented in three lectures at MSRI in spring 1985, we recommend the notes [1] of Akbulut and McCarthy. Other generalizations and extensions are noted in the bibliography.

Shortly thereafter, Cliff Taubes found a highly suggestive analytic interpretation [12] of Casson's $\lambda(\Sigma^3)$. Let Σ^3 be an oriented $\mathbb{Z}HS$, and consider the space \mathcal{A} of all connections on a fixed (trivial) $SU(2)$ -bundle P over Σ^3 . The group \mathcal{G} of gauge transformations of P is isomorphic to the group of all maps from Σ^3 into $SO(3)$. Then the tangent space to \mathcal{A} at A is the space $\Omega^1(\Sigma^3, \mathfrak{g})$ of 1-forms on Σ^3 taking values in the bundle of Lie algebras associated to P . The curvature $F(A)$ of A is an exterior 2-form with values in \mathfrak{g} , an element of $\Omega^2(\Sigma^3, \mathfrak{g})$. Using the invariant inner product on the Lie algebra of $SU(2)$ and a volume form on Σ^3 , the spaces $T_A(\mathcal{A}) = \Omega^1(\Sigma^3, \mathfrak{g})$ and $\Omega^2(\Sigma^3, \mathfrak{g})$ are canonically dual. Thus the operation assigning to a connection $A \in \mathcal{A}$ its curvature $F(A) \in \Omega^2(\Sigma^3, \mathfrak{g})$ is a 1-form Φ on \mathcal{A} . Furthermore, \mathcal{G} acts on \mathcal{A} preserving Φ . The zeros of Φ evidently correspond to flat connections. \mathcal{G} -orbits of flat connections correspond to equivalence classes of

representations of the fundamental group into $SU(2)$. Taubes defines a Poincaré-Hopf index of the 1-form Φ on \mathcal{A} and shows that the resulting Euler characteristic equals Casson's invariant.

This invariant depends on the Fredholm structure of the moduli space of flat $SU(2)$ -connections on Σ^3 . The “instanton homology” (see [6]) invented by Floer is a homology theory associated to Σ^3 whose Euler characteristic equals $\lambda(\Sigma^3)$. However, the Floer homology has been rather difficult to compute except in the simplest examples, while $\lambda(\Sigma^3)$ can be computed simply in terms of a surgery formula discovered by Casson. This remarkable formula is the key to calculating Casson's invariant. If $K \subset M$ is a knot in a $\mathbb{Z}HS$, and $K_{1/n}$ denotes $1/n$ -Dehn surgery on M along K (so that $K_{1/n}$ remains a $\mathbb{Z}HS$ and $K_{1/0} = M$; *see below*), then

$$\lambda(K_{1/n}) = \lambda(M) + n\Delta_K''(1)$$

where Δ_K'' denotes the second derivative of the Alexander polynomial Δ_K of K in M .

Given the power of this invariant, it was natural to try to extend it in various directions. The book under review describes an extension of $\lambda(\Sigma^3)$ to the case when Σ^3 is a $\mathbb{Q}HS$. If $K \subset S^3$ is a knot, then its “complement” is a compact 3-manifold N^3 with boundary a 2-torus (whose interior is homeomorphic to $S^3 - K$). One obtains a rational homology sphere $N_{p/q}$ by attaching to M^3 a solid torus $T \approx S^1 \times D^2$ by gluing the meridian $\{s\} \times \partial D^2$ of T to a (p, q) -curve on ∂N^3 . A (p, q) -curve on ∂N^3 is defined as any curve in the homology class corresponding to $(p, q) \in \mathbb{Z} \oplus \mathbb{Z} \cong H_1(\partial N)$ which is the linear combination: p times a generator $(1, 0)$ of the image of

$$\mathbb{Z} \cong H_1(N) \hookrightarrow H_1(\partial N)$$

plus q times an element $(0, 1)$ projecting to a generator under

$$H_1(\partial N) \rightarrow H_1(\partial N)/H_1(N) \cong \mathbb{Z}.$$

In particular, $N_{1/q}$ is a $\mathbb{Z}HS$ for each $q \in \mathbb{Z}$ while *every* $N_{p/q}$ is a $\mathbb{Q}HS$.

Casson's original definition and Walker's generalization each involve the moduli space $\text{Hom}(\pi_1(\Sigma^3), SU(2))/SU(2)$ of equivalence classes of representations of the fundamental group $\pi_1(\Sigma)$ into $SU(2)$. The representations themselves form a real algebraic set upon which the inner automorphisms of $SU(2)$ act algebraically. Accordingly, the moduli space is a real analytic space and stratifies into analytic submanifolds. Roughly, Casson's invariant counts—*algebraically*—the points in $\text{Hom}(\pi_1(\Sigma^3), SU(2))/SU(2)$ where Σ^3 is a closed oriented $\mathbb{Z}HS$.

The actual counting is a bit more involved. Following Casson, one decomposes the 3-manifold Σ^3 as a union of two solid handlebodies $W_1 \cup_F W_2$ of genus $g \geq 1$ along a closed surface F and computes the space $\text{Hom}(\pi_1(\Sigma^3), SU(2))/SU(2)$ in terms of this Heegaard decomposition. The space of interest $\text{Hom}(\pi_1(\Sigma^3), SU(2))/SU(2)$ is embedded in the larger space

$$X = \text{Hom}(\pi_1(F), SU(2))/SU(2)$$

corresponding to the surface $F \subset \Sigma^3$ under the natural restriction map, but there is considerably more space (and geometry) in X to study $\text{Hom}(\pi_1(\Sigma^3), SU(2))/SU(2)$. Then the space of equivalence class of representations

$$\text{Hom}(\pi_1(\Sigma^3), SU(2))/SU(2) \subset X$$

(alternately, the space of gauge-equivalence classes of flat connections) is the intersection of the images Q_i of

$$\mathrm{Hom}(\pi_1(W_i), \mathrm{SU}(2))/\mathrm{SU}(2) \hookrightarrow X.$$

The orientation of Σ^3 induces orientations of $W_1, W_2, F \subset \Sigma^3$, which in turn induce orientations on the various spaces $Q_1, Q_2 \subset X$, and Casson's invariant $\lambda(\Sigma^3)$ is a kind of oriented intersection number $Q_1 \cdot Q_2$.

To develop intuition for the nature of this invariant, consider the following simple analogue. Replace $\mathrm{SU}(2)$ by the circle group $\mathrm{U}(1)$. If Σ^3 is a $\mathbb{Z}HS$, every representation $\pi_1(\Sigma^3) \rightarrow \mathrm{U}(1)$ is trivial, but for a more general $\mathbb{Q}HS$ (say, one with cyclic fundamental group \mathbb{Z}_m), nontrivial representations $\pi_1(\Sigma^3) \rightarrow \mathrm{U}(1)$ will exist. For the purposes of this discussion (where only the abelianization of the fundamental group matters), it suffices to consider the case that Σ^3 is a *lens space*: glue two solid tori W_1 and W_2 along a common bounding 2-torus T . The inclusions of the boundary $T \rightarrow \partial W_j \subset W_j$ induce homomorphisms

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} &\cong \pi_1(T) \rightarrow \pi_1(W_j) \cong \mathbb{Z}, \\ (x, y) &\mapsto a_j x + b_j y \end{aligned}$$

where a_j and b_j are relatively prime integers. Then the analogue X' of X is the torus $\mathrm{Hom}(\pi_1(T), \mathrm{U}(1))$ dual to $\pi_1(T)$. Explicitly, X' consists of representations of the form

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} &\rightarrow \mathrm{U}(1) \\ (x, y) &\mapsto e^{i(\alpha x + \beta y)} \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. The analogues of the Q_j are the circles Q'_j (one-parameter subgroups with respect to the structure of X as a group) defined by

$$\alpha = t a_j, \quad \beta = t b_j.$$

All of the intersections of Q'_1 and Q'_2 are transverse and have the same oriented intersection number. The cardinality of $Q'_1 \cap Q'_2$ is the determinant $m = a_1 b_2 - a_2 b_1$, the order of $\pi_1(\Sigma^3)$. Thus $\# \pi_1(\Sigma^3)$ may be regarded as an "abelian" analogue of Casson's invariant.

The book under review is a monograph describing Walker's extension of Casson's invariant to $\mathbb{Q}HS$. This was Walker's 1989 Berkeley doctoral thesis, and a summary of this work may be found in his research announcement [13]. The idea of Walker's generalization is similar to Casson's definition: sum local expressions over the intersections of Q_1 and Q_2 in X , after perhaps making a generic perturbation to make them transverse. However, the presence of singularities in these spaces produces significant technical problems. Using the symplectic geometry of X (in which the Q_j are Lagrangian), Walker defines local invariants which give a meaningful invariant. Section 1 is background information on the moduli spaces of representations and its symplectic geometry. Section 2 contains the definition of $\lambda(\Sigma^3)$, and §3 develops its elementary properties. Many of these were proved by Casson in $\mathbb{Z}HS$ case. Section 4 proves the surgery formula: how $\lambda(\Sigma^3)$ behaves when surgery is performed on a knot in a $\mathbb{Q}HS$. Conversely, uniqueness of Walker's

invariant is proved in §5: any function of rational homology spheres vanishing on S^3 and satisfying the surgery formula must equal $\lambda(\Sigma^3)$. Section 6 contains various consequences of this theory. For example, Walker's λ is additive under connected sum and relates to the Rohlin invariant of a spin $\mathbb{Q}HS$. Two appendices summarize results concerning Dedekind sums and Alexander polynomials which are needed in the main body of the monograph.

This is a fascinating subject, and Walker's book is informative and well written. Together with [1] it makes a rather pleasant introduction to a very active area in geometric topology.

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