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BOOK REVIEW

Distributions and convolution equations, by S. G. Gindikin and L. R. Volevich. Gordon and Breach, Philadelphia, 1992, xi + 465 pp., \$95.00. ISBN 2-88124-753-9

The history of partial differential equations (PDE) is very long. The beginning can be traced back to the middle of the eighteenth century, when D'Alembert, Euler, and Bernoulli studied the equation of a vibrating string. This was just an example of a PDE but a very important one, coming as it did from mathematical physics. For this reason it was also very exciting, since one could easily observe the correspondence between theory and nature. Already this very simple (by modern standards) example stimulated a discussion between D'Alembert and Euler about the scope of functions to be considered; D'Alembert felt that only analytic functions should be considered, whereas Euler advocated the use of more general functions, since the string can really take any form (not necessarily analytic). This discussion influenced the further development of function theory. Other important equations (the Laplace equation and the 2- and 3-dimensional wave equations) also appeared in papers by D'Alembert and Euler, but only particular solutions were considered.

At the beginning of the nineteenth century important new equations taken from mathematical physics were studied more thoroughly (the Laplace and Poisson equations and the heat equation, which was first introduced by Fourier), but also a glimpse of general theory appeared (the potential theory of Green and Gauss). Then Cauchy was one of the first to value a general theory by itself, independently of applications. This produced the ideas behind the Cauchy-Kovalevskaya theorem—one of the first general theorems about PDE.

When nonanalytic functions were accepted by mathematicians in the second half of the nineteenth century, it was natural to try to pass from analytic to more general functions (e.g., smooth ones) in the general Cauchy problem. But Hadamard was the first to realize that this was absolutely nontrivial. Early important achievements to hyperbolic equations were obtained by Hadamard himself, but the first really significant progress for more general equations was due to Petrovsky.

In 1938 Petrovsky considered the Cauchy problem for a general system of PDE with constant coefficients

$$(1) \qquad \frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{j=1}^{N} \sum A_{ij}^{(k_0,\dots,k_n)} \frac{\partial^{k_0+\cdots+k_n} u_j}{\partial t^{k_0} \cdots \partial x_n^{k_n}}, \qquad k_0 < n_j,\ i = 1, \dots, N$$

with the initial conditions

$$\frac{\partial^l u_i}{\partial t^l}|_{t=0} = \phi_{il}(x), \qquad i = 1, \ldots, N, \; l = 0, \ldots, n_i - 1. \tag{2}$$

(Actually, Petrovsky also considered equations with coefficients depending on t, but there the result is not as effective, so we will discuss only this simpler case.) The main result of Petrovsky for the Cauchy problem (1)–(2) is as follows: This problem is correct in classes C^k if and only if the following condition (A) is satisfied: the imaginary parts of the roots (with respect to λ) of the algebraic equation

$$\det \left\| (i\lambda)^{n_i}\delta_{ij} - \sum_{k_0,\ldots,k_n} A_{ij}^{(k_0,\ldots,k_n)}(i\lambda)^{k_0}(i\xi_1)^{k_1}\cdots(i\xi_n)^{k_n} \right\| = 0 \tag{A}$$

are uniformly (in ξ) bounded below by a constant. In later terminology such systems are called correct in the sense of Petrovsky.

There are important classes of correct (in the Petrovsky sense) equations (or systems) where the theory can be advanced much further. The best-known examples are (strictly) hyperbolic and parabolic systems. Then variable coefficients can also be considered, and some important qualitative information can be obtained. However, it is always tempting for a mathematician to advance the theory to a maximal generality.

After creation of the theory of distributions in papers by S. Sobolev and L. Schwartz, a second wind came to the general theory of PDE. The theory of distributions was really the language that was needed to work effectively in many domains of PDE.

It is well known how important it is to consider not only smooth solutions but distribution solutions as well. The reason is the important role of the fundamental solutions (or Green functions) for constant coefficient equations and their Cauchy problems. And it so happens that the easiest way to describe fundamental solutions is to consider them as distributions.

In 1950–51 L. Schwartz reformulated Petrovsky results in the language of distributions. He also considered general convolution operators instead of derivatives in (1). In this way the Cauchy problem (1)–(2) can be reformulated for the general convolution equations. The simplest way is not to try to find a proper replacement for the initial conditions (2) but to switch to considering the equation with the right-hand side which vanishes in the half-space $t < 0$ and to look for solutions which also vanish there (this is the so-called "homogeneous Cauchy problem").

The next important development in the general theory of linear PDE is the invention of pseudodifferential operators. They present a natural substitute for convolution operators when the coefficients are variable, and in fact they allow us to work with a function and its Fourier transform simultaneously (hence the term "microlocal analysis"). The pseudodifferential operators also provide a tool to pass from some solvability results for the constant coefficient case to variable coefficients, provided the coefficients are sufficiently smooth and some reasonable uniformity for the constant coefficient problems (with coefficients "frozen" at an arbitrary point) is guaranteed.

Now we come to the main goal of the authors of the reviewed book: to give an up-to-date interpretation of the Petrovsky classical work. This includes, in particular, proper generalizations of condition (A) to convolution equations; extension of existence, uniqueness, and correctness results to a much wider spectrum of function spaces; and passing from convolution operators (which are analogues of differential operators with constant coefficients) to pseudodifferential operators which already generalize differential operators with variable coefficients. The authors pursue this goal with a rare thoroughness; they discuss as many function spaces as possible in the chosen approach.

In Chapter 1 convolution equations are investigated in spaces of strongly decreasing and slowly increasing functions and distributions, starting with the Schwartz space and then passing to weighted C^k and Sobolev spaces (with power weights possibly multiplied by an exponent $\exp\langle\omega, x\rangle$). The most important idea here is that the solvability of a convolution equation in function spaces is usually equivalent to the existence of the fundamental solution which is a "convolutor" (in corresponding spaces) or (equivalently) has a Fourier transform which is a multiplier in the Fourier transformed spaces. Therefore, in this and subsequent chapters the authors try to make a complete description of convolutors and multipliers in all possible scales of function spaces. Also, Schwartz-type kernel theorems are proved in many different function spaces.

Chapter 2 treats the homogeneous Cauchy problem for convolution equations in spaces of strongly decreasing and slowly increasing functions and distributions. After Fourier transform this amounts to considering spaces with some appropriate analyticity conditions. In Chapter 3 the spaces of exponentially decreasing and increasing functions and distributions are considered, which leads to a notion of exponential correctness (different from the notion of Petrovsky correctness).

Chapter 4 treats a nonhomogeneous Cauchy problem for convolution equations. For differential equations this problem can be treated if we just extend the solution to the half-space $t < 0$ by 0 and then apply the operator; the Cauchy data ($\phi_{il}(x)$ in (2)) emerge then in the right-hand side of the equation tensored with the Dirac δ-function and its derivatives. Therefore, we arrive at the necessity of studying the equation in the whole space but in spaces of distributions which are more smooth when $t > 0$. This is exactly what the authors do in this chapter.

Chapter 5 treats at last the "variable coefficients" case, i.e., the Cauchy problem for pseudodifferential equations. Here the general philosophy of the authors pays off; the thorough study of the constant coefficient case made in Chapters 1–4 allows pretty smooth transition to variable coefficients. The same spaces and types of problems as in previous chapters are considered here, and the only difference is that now the solvability conditions are only sufficient (and not necessary).

The last chapter, Chapter 6, is about the Wiener-Hopf equations. Here the difference with the standard convolution case is in fact only in the choice of spaces, but this difference is significant. The operators act from a subspace to a quotient space of usual function spaces, hence the necessity of a factorization of symbols, which is a difficult problem by itself. The corresponding results extend the classical work by M. Krein and many other authors. The real difficulty here is an instability of the factorization, so variable coefficients are not considered.

The strong point of the book is that it treats the convolution equations with an exhaustive thoroughness which was without doubt difficult to achieve. So I definitely recommend the book to the experts who work in this area. Most parts of it might also be useful for graduate students to supplement an advanced PDE course.

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