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BOOK REVIEW

Multiplication of distributions and applications to partial differential equations, by M. Oberguggenberger, Pitman Research Notes in Mathematics Series, vol. 259, Longman Scientific & Technical, Harlow, 1992, xvii+312 pp., $39.00. ISBN 0-582-08733-3

Multiplication of distributions and, in fact, various *nonlinear* operations on distributions or generalized functions happen to have quite a history. Some of the early sources of interest were in physics, where the Dirac delta-function δ has brought with it both convenient calculation methods and puzzlement about their rigorous justification. For instance, in quantum mechanics one would like to play with formulas such as

$$\delta \cdot (1/x) = -\delta'/2,$$
$$\delta^2 - (1/x)^2/\pi^2 = -(1/x^2)\pi^2,$$
$$(\delta_+)^2 = -\delta'/4\pi i - (1/x^2)/4\pi^2,$$
$$(\delta_-)^2 = \delta'/4\pi i - (1/x^2)/4\pi^2,$$

where $\delta_+ = (\delta + (1/x)/\pi i)/2$ and $\delta_- = (\delta - (1/x)/\pi i)/2$; see [G-DS, M] or pages 18–20 of the book under review. However, such formulas could hardly be justified, except for certain rather ad hoc and questionable computational manipulations.

Another source of interest came from *nonlinear shock waves.* Indeed, even in the case of the basic equation

$$U_t(t,x) + U(t,x)U_x(t,x) = 0, \qquad t \geq 0,\ x \in \mathbb{R},$$

some of the simplest physically relevant solutions are given by the Riemann solvers

$$U(t,x) = u_l + (u_r - u_l)H(x - x_0 - st), \qquad t \geq 0,\ x \in \mathbb{R},$$

where $x_0, u_l, u_r \in \mathbb{R}$, $u_l > u_r$, $s = (u_l + u_r)/2$, while H is the Heaviside function. Yet such a solution is *no longer classical*, which in this case would of course mean that $U \in \mathcal{C}^1(\Omega)$, with $\Omega = [0,\infty) \times \mathbb{R}$. Now, if we tried—as physicists or engineers may often be tempted—to replace U given by the Riemann solver in the above equation, then the *nonlinear convective term* $U \cdot U_x$, so typical for many equations of physics, would immediately lead a mathematician to the problem of having to make sense of the *product* $H \cdot \delta$, a product in which H is *discontinuous*, while δ is *not* a usual function.

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Here, however, we face two different kinds of difficulty. First, as we shall see in a moment, it is *not* at all *trivial* to build a good enough multiplication theory for generalized functions. Second, the multiplication of given generalized functions may present what can appear as a disturbingly *large variety* of possible outcomes. This latter point is quite well illustrated by the simple fact that the *weak* or generalized solutions of the type of the Riemann solvers are *different* for the shock wave equation and

$$(U^m)_t + U \cdot (U^m)_x = 0, \qquad t \geq 0,\ x \in \mathbb{R},$$

where $m \geq 2$. And this may appear rather surprising in view of the fact that, when it comes to their *classical*—that is, $\mathcal{C}^1$-smooth—solutions, the two mentioned equations will have the same solutions.

The first difficulty above became obvious early, although for quite a while it suffered from various misinterpretations. Indeed, back in 1954 in [Sc] it was shown that one could not define a distribution multiplication without losing some of the usual properties of the product when restricted to insufficiently smooth—for instance, only $\mathcal{C}^0$-smooth—functions; for details see pages 27–36 of the book under review or [R4, pp. 1–9]. Unfortunately, however, that fact has often been misinterpreted as, for instance, in the following citation: "It has been proved by Schwartz... that an associative multiplication of two arbitrary distributions cannot be defined," [H, p. 9].

That first difficulty was not helped by the 1957 negative result in [L], which showed that the *linear* first-order PDE

$$U_{x_1} + iU_{x_2} - 2i(x_1 + ix_2)U_{x_3} = f(x), \qquad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

did not have distribution solutions in any neighbourhood of any point $x \in \mathbb{R}^3$, for f ranging over a large subset of $\mathcal{C}^\infty(\mathbb{R}^3)$.

In spite of such a state of affairs, starting with the late sixties, a massive theoretical effort got under way in order to find distributions or generalized solutions for large classes of nonlinear PDEs; see [Li] for one of the first such contributions. However, this effort has been undertaken without the existence of a *comprehensive* nonlinear theory of generalized functions. Instead, various functional analytic methods have been used, with the risk—long unnoticed—that the respective solution concepts may have been insufficiently clarified; for details see [R4, pp. 30–43].

A first comprehensive nonlinear theory of generalized functions was presented in [R1–R4], based on an *algebraic* rather than functional analytic approach. Not much later in [C1–C3] a particular case of that theory was introduced and quite significantly developed in an independent way. Another stage was reached in [B], where *numerical* applications to nonlinear PDEs have been reported, showing once again the relevance, both theoretical and applicative, of the recent nonlinear theories of generalized functions. In this way since the late seventies there has been an ongoing development in such comprehensive nonlinear theories, the results being published in several dozen papers in addition to the books mentioned. One of the main achievements so far has been the solution of large classes of linear and nonlinear PDEs that earlier were unsolved or proved to be unsolvable within distributions or hyperfunctions.

The book under review comes as the latest important published contribution to the development of an *algebra* based nonlinear theory of generalized functions. The author's strong background in analysis, functional analysis, and nonlinear PDEs is quite clearly manifest in the book under review. In addition, he shows a deep understanding of a large variety of approaches to generalized functions, including the algebraic approach.

In this vein the book starts in the preface with an account of the essence of the problems faced by any nonlinear theory of generalized functions—namely, to deal with "... mathematical models that involve

(a) nonlinear operations,
(b) differentiation, and
(c) the presence of singular objects, like measures or nondifferentiable functions...".

The author further points out that "classical nonlinear analysis can certainly deal with (a) and (b), while the theory of distributions has proved to be successful in handling (b) and (c) simultaneously...".

It remains, therefore, for any *nonlinear* theory of generalized functions to deal with (a), (b), and (c) *at the same time*.

Let us now review the main contributions of the book, which gives a well-selected, well-written, and rather concise overview of the subject and presents a wealth of connections with classical, more particular approaches. The various topics are presented with an original and particularly valuable insight, which in the words of the author tries—and in our view, clearly succeeds—"to emphasize ideas rather than generality". Among them let us mention the following. In §§6–8 a good review of "intrinsic products" is presented. The sheaf structure of Colombeau's algebra $\mathcal{G}(\Omega)$ of generalized functions is detailed in §9, while §12 gives a number of variants of $\mathcal{G}(\Omega)$ which are useful in solving different classes of linear and nonlinear PDEs.

Coming to nonlinear PDEs, semilinear and quasilinear hyperbolic systems are presented in §§13–20 in the context of generalized solutions. This presentation, among others, can serve as a good introduction to the respective subjects. In particular in §§19 and 20 rather general, possibly *nonconservative* nonlinear PDEs and their weak solutions are studied.

A concise account of the general algebraic theory is presented in §§21 and 22, followed by *nonstandard* approaches to generalized functions, to which §23 gives a good introduction. Indeed, the discourse in §23 is kept focused on the basic structures and facts without technical complications, such as those related to ultrapowers. Instead, the natural sequential approach, pioneered by Schmieden and Laugwitz [SL], is given its proper emphasis.

The book also contains a number of original contributions of the author. Among those already published elsewhere we mention Proposition 7.6 on page 63, which deals with the Fourier and strict product defined for certain distributions. Examples (d) and (e) on pages 70–72 highlight certain properties of such or related products. Theorems 15.1 and 15.2 on pages 137–145 give rather powerful results on delta-waves for general hyperbolic systems in two independent variables. A whole set of existence and uniqueness results in Colombeau's sense can be found in §§16–20. Finally, §24 on delta wave solutions of hyperbolic systems is another of the author's previously published original contributions.

Concerning the author's original results not published elsewhere, we mention the duality method in §5 and Proposition 7.8 on page 65. The same holds for the alternative proof of Theorem 15.2 on pages 143–145. Finally, the regularity theory of §25 also belongs to the author's list of new and unpublished original contributions.

In conclusion, this book is highly recommended both for first-time readers in generalized functions and for those who enjoy a certain familiarity with the subject. For both categories of readers a main attraction of the book is its focus on ideas rather than on generality. Also, it presents an impressive array of concepts, results, and applications, which include many of the most basic nonlinear PDEs. For the reader more familiar with the subject, the depth of insight and analysis at various places in the book can only be an additional incentive and reward.

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