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Matroid theory, by James G. Oxley. Oxford University Press, London 1992, xi + 532 pp., $79.00. ISBN 0-19-853563-5

On one level, matroid theory is just a combinatorial abstraction of linear algebra. We can define a matroid as a set E together with a collection $\mathscr{B}$ of finite subsets of E called *bases* such that:

(1) $\mathscr{B} \neq \varnothing$;
(2) if $X, Y \in \mathscr{B}$ and $X \subseteq Y$, then $X = Y$;
(3) if $X, Y \in \mathscr{B}$ and $x \in X$, then there exists $y \in Y$ such that $(X - \{x\}) \cup \{y\} \in \mathscr{B}$.

It is easy to see that the bases of a finite-dimensional vector space satisfy these axioms. In fact, more generally, the bases contained in any spanning subset of a finite-dimensional vector space satisfy them as well, giving rise to vector matroids. Likewise, the spanning trees of a finite connected graph satisfy these axioms (where E is the set of edges of the graph), and the resulting matroids are known as graphic matroids. So do the transcendence bases of a field extension of finite transcendence degree (or any "spanning" subset of such an extension), giving us algebraic matroids. We can also deal with infinite bases at the cost of an additional axiom in order to get a reasonably interesting theory.

On another level, however, a large part of the fascination of matroid theory is that so many different concepts from linear algebra, and from graph theory, have analogues in the theory. Thus, independent sets, dependent sets, spanning sets, dimension, the span operator, subspaces, hyperplanes (or subspaces of codimension one), and the lattice of subspaces all have analogues in matroid theory, as do circuits (simple closed paths) and bonds (minimal edge cut-sets) from graph theory. Even more amazing is that each of these concepts may be taken as the starting point for the theory, given an axiomatization, and used to define all of the other concepts. For example, hyperplanes are maximal sets containing no basis, whereas circuits are minimal sets not contained in a basis.

A similar situation exists for topological spaces, where one may start with open sets, closed sets, closure operator, etc. Matroids must surely hold the record in mathematics for the largest number of equivalent axiomatizations, at last count some 57 varieties, each in terms of one of 13 distinct concepts from the theory.

This versatility contributes greatly to the usefulness of matroid theory in a number of branches of combinatorics, discrete geometry, and algebra. Not the least of these branches is combinatorial optimization, where the existence of a certain kind of greedy algorithm guarantees that a matroid is lurking in the background. Several structures closely related to matroids, such as greedoids, oriented matroids, and submodular functions are also very commonly employed in combinatorial optimization. Other areas of application include structural rigidity, electrical engineering, graph theory, and stratification of the Grassmannian.

Matroids were invented by Hassler Whitney in the mid-1930s, followed immediately by Garrett Birkhoff's work on their lattice-theoretic equivalent, geometric lattices. Some of the deepest work in the field was done in the 1950s by W. T. Tutte, who gave an excluded minor characterization of several important classes of matroids, including the regular or unimodular matroids. These matroids are defined as those which may be represented as a vector matroid over every field.

Matroids enjoyed an explosion of interest in the 1970s and 1980s. This interest now has somewhat lessened and been replaced partially by the current interest in oriented matroids, a variation in which each (ordered) basis is given a sign indicating its orientation. A prototypical example would be any subset of 3-dimensional real affine space, in which the bases are the affinely independent subsets of cardinality four, and the orientation of such a basis depends on whether it forms a right- or left-handed tetrahedron. Oriented matroids have a wide variety of applications from convex polytopes to Pontrjagin classes.

One particular area of the original (unoriented) matroid theory has remained very active to the present day. This area may be called decomposition theory. The concept of k-connectedness from graph theory has an analogue in matroid theory, and a *k-sum* of two matroids is a particular construction which combines two k-connected matroids so that the resulting matroid is not $(k+1)$-connected. Decomposition theory describes how the matroids in a given class of matroids must decompose under k-sums for certain values of k. Decomposition theory frequently ties in with the older excluded minor results. Paul Seymour initiated this branch of matroid theory, primarily with his celebrated theorem on the decomposition of regular matroids. This theorem states that every regular matroid decomposes via 1-sums, 2-sums, and 3-sums into matroids that are either graphic matroids, matroid duals of graphic matroids, or one particular matroid known as R_{10}. Among the consequences of this theorem was the first polynomial-time algorithm for determining whether a given matrix is totally unimodular. One of several workers who remain active in decomposition theory is the author of the book under review.

There are already two standard and fairly complete references in matroid theory, namely, Welsh's volume and the Cambridge University Press series edited by this reviewer. What then is the usefulness of Oxley's book? First, it covers some topics not covered in either of the older works. This includes primarily the background for decomposition theory, such as an entire chapter on 3-connectedness, and thorough coverage of splitters. Second, it includes self-contained proofs of some of the major excluded-minor characterizations of hereditary matroid classes: namely, regular matroids, ternary matroids, and graphic matroids. These are more accessible proofs which were not available when the earlier volumes were written. Third, Oxley's book is more easily used as a graduate textbook than the other two. It includes more background, such as finite fields and finite projective and affine geometries, and the level of the exercises is well suited to graduate students.

The book is well written and includes a couple of nice touches. One is an appendix compiling "interesting matroids" and some of their properties. The other is a whole chapter devoted to currently unsolved problems. To cover all of this in a single volume, the author obviously had to omit some topics; these are already well covered in the older volumes and not essential for an introduction. These topics include most of the connections with combinatorial optimization, nonbipartite matching, strong and weak maps, and the matroid invariants such as the characteristic polynomial and Whitney numbers.

I would not recommend the book for those whose main interest comes from combinatorial optimization. In any other case, however, this is a very useful book. I recommend it highly both as an introduction to matroid theory and as a reference work for those already seriously interested in the subject, whether for its own sake or for its applications to other fields.

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