which has been proved, is invoked, while what is actually needed is the clean intersection calculus, which has not.

But the above are quibbles; *Fourier integrals in classical analysis* rewards the reader with a thorough account of some of the last decade's most important developments in Fourier analysis, many of them due to its author. It belongs on the bookshelf of anyone seriously interested in the subject.

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Introduction to the general theory of singular perturbations, by S. A. Lomov. Translations of Mathematical Monographs, vol. 112, American Mathematical Society, Providence, RI, 1992, xviii + 375 pp., $201.00. ISBN 0-8218-4569-1

The importance of the concept of "perturbation" has long been recognized in the field of celestial mechanics. In particular, the study of the motion of, say, a planet under the gravitational influence of the sun leads to the question of how that motion is perturbed by the presence of a second planet. These questions have had profound consequences: for example, the discovery of new planets (e.g., Neptune) and the investigation of new physical theories (e.g., general relativity). More generally, perturbations give a relationship between two (actually a family of) problems, one of which may be more tractable to quantitative analysis. This relationship can be formalized using the machinery developed by Poincaré: series in powers of a small parameter that may be convergent or merely "asymptotic", that is, increasingly accurate for decreasing values of the parameter.

Often in the use of perturbation methods singularities of various types arise, and it has become customary to speak of "singular" perturbation problems. In general, the singularity involves the failure of the perturbed problem to converge uniformly to the solution of the unperturbed problem as the small parameter goes to zero. For example, the nonuniformity can appear as a rapid transition in function values ("layer structure") or as an increasingly rapid oscillation. The seminal work in this area was done by Prandtl [8], who studied the motion of an incompressible fluid past an object for large values of the Reynolds number. He observed the rapid change in the velocity of the fluid near the boundary of the object ("boundary layer"). Through the course of the twentieth century, the concept of a singular perturbation has been found useful in many fields and has led to an explosion of interest in these methods, both by researchers in the field and by mathematicians. Robert O'Malley's book review [7] gives a concise account of the history of singular perturbations with many references.

The efforts of scientists to construct useful approximations have generated a great variety of techniques for solving singular perturbation problems. At the same time, mathematicians have been presented with some very challenging questions concerning the validity and accuracy of the approximations. A popular method for analyzing problems exhibiting layers has been the patching of approximations obtained inside and outside the layer to construct a uniform approximation. This patching is accomplished through various formal "matching" techniques (see Van Dyke [10], Erdélyi [3], Eckhaus [2]). A successful competing approach involving layer corrections was worked out by O'Malley [6] and others. For oscillatory problems the multivariable method (discussed below) has been strongly promoted through the books of Kevorkian and Cole [4] and Nayfeh [5], while the method of averaging, developed in the former Soviet Union by Krylov and Bogolyubov (see Bogolyubov and Mitropol'skiĭ [1]), has been employed in a broad range of problems in nonlinear oscillations. Smith [9] gives a nice discussion of these methods, including a comparison of the results.

The present volume makes a remarkable attempt to initiate a general theory of singular perturbations based on a multivariable method, which in this context is called the method of "regularization". The book's thesis is that a single procedure can be used to obtain approximations to a wide variety of problems, including both layer-type and oscillatory problems, and that the approximations are uniformly valid for sufficiently small values of a parameter. (The issue of "how small" is not addressed here and, in fact, is considered by many workers in this field to belong to the area of bifurcation theory.)

The basic ingredients of the regularization method are already evident in the analysis of the linear Cauchy problem (Chapter 2)

$$L_\varepsilon \equiv \varepsilon u' - A(x)u = h(x), \qquad u(0, \varepsilon) = u^0,$$

where the matrix function $A(x)$ and the vector function $h(x)$ are smooth and defined on a finite interval $[0, a]$ and ε is a small positive parameter. In the simplest case it is also assumed that the eigenvalues $\lambda_i(x)$ of $A(x)$ are simple and not identically zero and that the real parts of the eigenvalues can be ordered for all x. Define

$$t_i = \frac{1}{\varepsilon}\int_0^x \lambda_i(\tau)\,d\tau, \qquad t = \{t_i\}.$$

If the t_i are treated as independent variables, then the extended function $\tilde{u}(x, t, \varepsilon)$ satisfies

$$T_\varepsilon \tilde{u} \equiv \varepsilon \frac{\partial \tilde{u}}{\partial x} + \sum_{i=1}^{n} \lambda_i(x) \frac{\partial \tilde{u}}{\partial t_i} - A(x)\tilde{u} = h(x)$$

and $u(0, 0, \varepsilon) = u^0$. Now one seeks a formal series solution

$$\tilde{u} = \sum_{i=0}^{\infty} u_i(x, t)\varepsilon^i$$

and obtains a sequence of problems for the coefficients:

$$T_0 u_0 = h(x), \qquad u_0(0, 0) = u^0,$$
$$T_0 u_i = \frac{\partial u_{i-1}}{\partial x}, \qquad u_i(0, 0) = 0,$$

where $i \geq 1$. These problems have infinitely many solutions, but experience with special cases suggests that only the "resonance-free" solutions are of interest. (Thus we reject "secular" terms like $t_i e^{t_i}$.) The space of resonance-free solutions is of the form

$$U = \left\{\sum u^{ij}(x)e^{t_i} + \sum u^i(x)\right\}.$$

One can show that there are unique coefficient functions u_i in U satisfying the above conditions and that the resulting series is an asymptotic series for a solution of the Cauchy problem. Moreover, in certain special cases the uniform convergence of the series for $0 \leq x \leq a$ can be established if ε is sufficiently small.

For each new problem the outline of the method remains the same, but there are several basic difficulties that must be resolved anew. First, one must decide what new independent variables to introduce. (These must contain all the possible singular behavior of the solution with respect to the parameter.) Second, an appropriate space of resonance-free solutions must be defined. Then a solvability theory for the set of equations in this space must be established. Finally, the formal approximation must be shown to be a good estimate of some solution of the original problem for small values of the parameter.

Despite the difficulties, the author is able to carry through this program for a nice variety of problems: linear boundary value problems for ordinary differential equations, certain linear integrodifferential equations, problems with rapidly oscillating coefficients, linear parabolic equations in one space variable with solutions periodic in time, and linear elliptic equations in a cylindrical domain. Conspicuously absent is any analysis of nonlinear problems beyond the nonlinear Cauchy problem. However, since the original Russian version appeared in 1981, some new problems have been solved by the regularization method, and several of these papers are referenced in the preface to the English edition.

This is a very attractive book for a number of reasons. The author has done an admirable job of explaining clearly and in great detail a rather complicated subject, and this careful exposition has been preserved by an unusually good translation. The expert in singular perturbations will find here a thorough rendition of the important work of a Russian school of mathematicians begun in the late 1950s. The novice will benefit from the introduction to the basic concepts and methods of perturbation theory, the numerous examples and comparisons of different techniques, and the detailed explanations. In addition, the work contains some interesting connections with other areas, especially the spectral theory of operators and Fourier analysis.

In the preface to the English edition the author states that a theory of singular perturbations "... is necessary to at least achieve a deeper understanding of how smoothness is preserved in the presence of degeneracy." It appears that in this book he has constructed the foundation for such an understanding.

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