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BOOK REVIEW

Products of groups, by B. Amberg, S. Franciosi, and F. de Giovanni. Oxford Mathematical Monographs, Clarendon Press, Oxford, 1992, xii+220 pp., $78.00. ISBN 0-19-853575-9

If A and B are subgroups of a (multiplicatively) written group G, the *product* AB of A and B is defined to be the subset of all elements of G with the form ab where $a \in A$ and $b \in B$. It is well known from elementary group theory that AB is a subgroup if and only if $AB = BA$, i.e., the subgroups A and B are *permutable*. Should it happen that AB coincides with the group G, with the result that $G = AB = BA$, then G is said to be *factorized* by its subgroups A and B. The factorization is called *proper* if both A and B are proper subgroups of G.

Perhaps the first question one might ask about factorizations is: which nontrivial groups have a proper factorization? Obvious examples of groups with no proper factorizations are cyclic p-groups and groups of Prüfer type p^∞. More formidable examples are (i) Tarski groups, i.e., insoluble groups whose proper subgroups are cyclic, and (ii) the Heineken-Mohamed groups [13]; these are nonnilpotent metabelian groups whose proper subgroups are subnormal and nilpotent.

On the other hand, by a well-known theorem of P. Hall every finite soluble group G is a product of pairwise permutable Sylow subgroups (arising from a Sylow basis). From this it is easy to deduce that if G has no proper factorizations, then it must be a cyclic p-group. (An extension to infinite groups is given in Chapter 1 of the book under review.)

Factorizations of finite (nonabelian) simple groups have been widely studied. Liebeck, Praeger, and Saxl [24] have determined all maximal factorizations, i.e., those in which both factors are maximal subgroups, of finite simple groups. From their work one can read off the list of finite simple groups with no proper factorizations; the unitary group $U_3(7)$ is an example.

Another natural path of inquiry opens up when one asks how the structure of the factors A and B affects the structure of a factorized group $G = AB$. Obviously, if A and B are finite, then G is finite and its order is given by

$$|G| = |A| \cdot |B|/|A \cap B|.$$

Thus a group which is the product of two finite p-groups is itself a finite p-group. Another property which carries over from the factors of a factorized group to the group itself is the property of being perfect, i.e., coinciding with the commutator subgroup. Nevertheless, this phenomenon seems to be quite rare.

Indeed, if one experiments with properties such as solubility, torsion, or of finite exponent, one soon realizes the difficulty of using the factorization to obtain information about the structure of the group. There are in fact some quite striking counterexamples:

(i) every countable group can be embedded in a group which is a product of two Tarski groups having all their proper nontrivial subgroups of prime order (Ivanov [18]);

(ii) there is a group which is the product of two locally finite subgroups and which has free subgroups of rank > 1 (Suchkov [31]).

Thus a group which is a product of torsion groups can have elements of infinite order.

The theory of factorized groups began with a series of articles by Zappa [40], Rédei [27], and Szép [34] in the 1940s and 1950s. The latter was especially interested in factorizations of finite simple groups. Then Huppert [16] and Cohn [9] considered groups that are expressible as products of cyclic groups, clearly a natural case at which to look. But it was a short note by Itô [17] in 1955 which set the theory of products in motion. Itô's result is:

Theorem. *If a group G is the product of two abelian subgroups A and B, then G is metabelian, i.e., it is soluble with derived length ≤ 2.*

The proof of this result is both short and elegant, relying only on some elementary facts about commutators. It can even be presented here.

Let a, $a_1 \in A$ and b, $b_1 \in B$. Since $G = AB = BA$, we can write $b^{a_1} = a_2b_2$ and $a^{b_1} = b_3a_3$ where $a_i \in A$ and $b_i \in B$. (Here x^y denotes the conjugate $y^{-1}xy$.) Then, writing the commutator $x^{-1}y^{-1}xy$ as $[x, y]$, we have

$$\begin{aligned}[a, b]^{a_1b_1} &= [a, b^{a_1}]^{b_1} = [a, a_2b_2]^{b_1} = [a, b_2]^{b_1} = [a^{b_1}, b_2] \\ &= [b_3a_3, b_2] = [a_3, b_2]\end{aligned}$$

since A and B are abelian. In the same way one finds that $[a, b]^{b_1a_1} = [a_3, b_2]$. It follows that $[a, b]$ and $[a_1^{-1}, b_1^{-1}]$ commute for all a, $a_1 \in A$ and b, $b_1 \in B$, i.e., the commutator subgroup $G' = [A, B]$ is abelian, and so G is metabelian.

Many attempts have been made to generalize this simple argument, but with scant success. It is not even known if a group which is the product of an abelian group and a nilpotent group of class 2 is soluble. At present the most satisfactory extension of Itô's result is:

Theorem (Černikov [7]). *A group which is the product of two centre-by-finite subgroups is soluble-by-finite.*

However, it is not known if a group which is a product of two abelian-by-finite subgroups is soluble-by-finite.

After the appearance of Itô's theorem attention shifted to finite groups that are products of a pair of nilpotent groups, the conjecture being that such groups are soluble. The motivation here was provided by the well-known Burnside $p - q$ Theorem: If p and q are primes, then a group of order $p^m q^n$ is soluble; equivalently, a product of two subgroups with prime power orders is soluble. The outcome of this line of investigation was the celebrated result of Kegel [19] and Wielandt [36].

Theorem. *A finite group which is the product of two nilpotent subgroups is soluble.*

It is not known if the derived length of the group can be bounded in terms of the nilpotent classes of the factors. Nor is it known if the Kegel-Wielandt Theorem can be extended to infinite groups.

One difficulty in working with infinite factorized groups is the lack of really convincing examples of factorizations. This has led to criticism that factorizations are "unnatural" in infinite group theory. The situation for finite groups is quite different, since one can often infer the existence of a factorization on purely arithmetic grounds. For example, a finite group is often expressible as the product of two nonconjugate maximal subgroups; indeed this is always true for finite soluble groups by a theorem of Ore [26]. One of the few natural examples of an infinite factorized group with nonnormal factors is

$$\mathrm{GL}_n(\mathbb{R}) = O_n T_n$$

where O_n is the group of real $n \times n$ orthogonal matrices and T_n is the group of real $n \times n$ upper triangular matrices with positive diagonal entries. This factorization is a consequence of the so-called QR-factorization of a matrix, itself a by-product of the Gram-Schmidt Procedure.

In 1961 Kegel introduced the notion of a *triple factorization*. This is a factorization of a group G involving three subgroups A, B, C of the type

$$G = AB = BC = CA.$$

The evidence is that the existence of a triple factorization can have greater consequences for the group structure than does a single factorization. For example, Kegel [20] proved:

Theorem. *A finite group which has a triple factorization by nilpotent subgroups is nilpotent.*

However, Kegel's result is definitely not true for infinite groups. This is shown by a nice construction due to Holt and Howlett [14] and Sysak [32]. One starts with a commutative radical ring R and then constructs a group G_R which has a triple factorization by abelian subgroups, one of them normal. By appropriate choice of the ring R it can be arranged that G_R is not nilpotent or even residually nilpotent.

On the other hand, quite recently Stonehewer and the reviewer [28] were able to show that a triple product of abelian groups does have some vestige of nilpotence.

Theorem. *If a group G is a triple product of abelian groups, then every chief factor of G is central.*

This means that G is a $\bar{Z}$-group in the terminology of Kuroš [21]. The theorem follows at once from a result about products of two abelian groups which had somehow escaped detection hitherto.

Theorem [28]. *Let the group G be the product of two abelian subgroups A and B. Then each chief factor of G is centralized by either A or B.*

The proof uses ring-theoretic techniques due to Howlett [15].

As has already been mentioned, nothing is known about a group which is a product of two soluble subgroups, and in particular it is unknown if the group is soluble. Because of this unsatisfactory situation, much of the research of the last twenty-five years has been concerned with the structure of *soluble* groups G with a factorization $G = AB$ where A and B have specific properties. This began with work of Seseken [30] and Amberg [1]. In fact all three authors of the book under review have been prominent in the development of the theory.

In the study of factorized soluble groups the concept of the factorizer of a normal abelian subgroup is frequently useful. Let N be a normal subgroup of a group $G = AB$ where A and B are subgroups. Then the *factorizer* of N is the subgroup

$$X(N) = AN \cap BN.$$

It is easy to see that

$$X(N) = (A \cap BN)N = (AN \cap B)N = (A \cap BN)(AN \cap B).$$

Thus $X(N)$ is a triple product in which the factor N is normal. In many cases it suffices to show that $X(N)$ has the property in question, so the problem reduces to one about a triply factorized group with one factor normal. Thus one can replace G by $X(N)$. When G is soluble, one would naturally want N to be abelian. Now $L = (A \cap N)(B \cap N)$ is normal in G in this case, and it may also be possible to pass to G/L, in which event one even has the situation

$$(*) \qquad G = AB = AN = BN, \qquad A \cap N = 1 = B \cap N.$$

Now N is a module over A, and A and B are complements of N in G. As is well known, there is a derivation $\delta : A \to N$ defined by the rule $aa^{\delta} \in B$, $(a \in A)$. Moreover, δ is surjective, since $G = AB$. Also, should A and B be conjugate, they will be equal and $G = A = B$.

It is already clear from these remarks that problems about factorized soluble groups are likely to involve both module theory and cohomology. As a case in point we mention a nice result of Lennox and Roseblade [23].

Theorem. *If the soluble group G is the product of two polycyclic subgroups A and B, then G is polycyclic.*

Arguing along the above lines, with N the penultimate term of the derived series of G, one reduces to the situation $(*)$. Also, using results from Roseblade's important theory of group rings of polycyclic groups [29], one can further assume that N is finite and minimal normal in G and that N is not centralized by the Fitting subgroup of A. A cohomological vanishing theorem of the reviewer then shows that $H^1(A, N) = 0$, so that A and B are conjugate and hence equal G.

This theorem has been a model for subsequent investigations of soluble groups that are factorized by subgroups having finite rank in some sense. The deepest results obtained so far are due to Sysak [33] and Wilson [38].

Theorem. *If a soluble group G is the product of subgroups A and B with finite abelian section rank, then G has finite abelian section rank.*

The proof of the theorem involves a delicate analysis of surjective derivations from a soluble group A with finite abelian section rank to an A-module N; the object is to show that the additive group of N has finite abelian section rank, for which purpose deep results of Brookes [6] on modules over groups with finite rank are used. In many ways this result is the culmination of the work of the last twenty-five years.

Triple factorizations of soluble groups by nilpotent subgroups have been considered by Amberg, Franciosi, and de Giovanni [3, 4, 5]; the object is to generalize Kegel's theorem under suitable restrictions of finite rank.

Other results of interest for a factorized soluble group $G = AB$ are: (i) if A and B are torsion, then so is G; and (ii) if A and B have finite exponent, then so does G (Černikov [8], Zaĭcev [39]).

The normal and subnormal structure of a factorized group have also been investigated. Suppose that a group G has a proper factorization $G = AB$. One may ask if there is a nontrivial normal subgroup of G contained in A or B. It was shown by Itô [17] that this is true if A and B are finite abelian groups. However, an example of Gillam [11] disproves the conjecture when A and B are finite p-groups. It is also false for A and B infinite abelian, but some sufficient conditions for its truth have been found by Franciosi and de Giovanni [10].

Dually one can ask if there is a proper normal subgroup of G containing A or B. This is true when A and B are finite nilpotent groups (Kegel [19]), but false if A and B are infinite elementary abelian p-groups (Holt and Howlett [14]). Again suitable finiteness restrictions imply the truth of the conjecture (Amberg, Franciosi, and de Giovanni [2]).

If H is a subnormal subgroup of $A \cap B$, one can ask if H is subnormal in $G = AB$. This has been proved for G finite by Wielandt [37] and Maier [25], while it is known to be false for A and B infinite abelian (Holt and Howlett [14]); again some sufficient conditions are known.

Finally, we mention that there has been some work on groups which are expressible as the product of a finite collection of pairwise permutable abelian subgroups $H_1, \ldots, H_n$:

$$G = H_1 H_2 \cdots H_n, \qquad H_i H_j = H_j H_i.$$

Tomkinson [35] showed that if each H_i is an abelian minimax group, then G is a soluble minimax group. A special case of interest due to Heineken and Lennox [12] states that if each H_i is finitely generated, then G is polycyclic. On the other hand, an example of Marconi exhibits a nontorsion group $G = ABC$ where A, B, C are pairwise permutable, locally cyclic torsion groups. Of course, every finite soluble group is a product of pairwise permutable nilpotent subgroups by the theorem of P. Hall. The existence of such factorizations in infinite soluble groups has been discussed by Lennox and the reviewer [22].

All of the above problems, and many more, receive a full discussion in the book under review. The emphasis is on infinite groups, although the proof of the Kegel-Wielandt theorem is given. The reader is expected to have a good basic knowledge of group theory, with particular reference to infinite soluble groups. However, the authors have tried to make the book self-contained. For example, they give a complete account of the cohomological vanishing theorems of the reviewer, which are used quite extensively.

The book is clearly written and well organized. It contains many interesting counterexamples and a full bibliography. Most of the material appears in book form for the first time. The reviewer's main regret is that the authors could not find space to include proofs of the results of Sysak and Wilson. It might have been better to have omitted some other material to make room. However, quibbles aside, the authors are to be congratulated on an excellent job. The book will be the standard reference for years to come.

The theory of products of groups is a small part of group theory, and it will not appeal to every group theorist. However, with its connections with ring theory, module theory and cohomology, and the many challenging open problems, it is an attractive area for further research.

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