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BOOK REVIEW

Schur's algorithm and several applications, by M. Bakonyi and T. Constantinescu. Pitman Research Notes in Mathematics Series, volume 261, Longman Scientific and Technical, Harlow, 1992, 190 pp., $54.95. ISBN 0-582-90120-9 Copublished in the U. S. by John Wiley & Sons. ISBN 0-470-21974-2

Operator methods in interpolation theory have deep classical roots which retain great importance for the modern subject. The sources for the book by Bakonyi and Constantinescu are (1) Schur's study [33] of power series which represent analytic functions which are bounded by one in the unit disk and (2) Szegö's theory [34] of orthogonal polynomials on the unit circle. The subject has evolved through a series of generalizations which are formulated in the language of Hilbert space operators, and today it is an active area with important engineering applications. All of these elements, classical and modern, are represented in the book.

Let S be the *Schur class* of analytic functions which are bounded by one on the unit disk D of the complex plane. Given $f(z)$ in S, define a sequence $f_0(z), f_1(z), \dots$ in S by $f_0(z) = f(z)$ and

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f}_n(0) f_n(z))}, \qquad n \geq 0.$$

If $|f_r(0)| = 1$ for some r, then $f_r(z)$ is a constant, and we take $f_k(z) \equiv 0$ for all $k > r$. This occurs if and only if $f(z)$ is a Blaschke product of r factors; this means that,

$$f(z) = C \frac{z - a_1}{1 - \bar{a}_1 z} \cdots \frac{z - a_r}{1 - \bar{a}_r z},$$

where $a_1, \dots, a_r$ are points in D and $|C| = 1$. The construction establishes a one-to-one correspondence between S and the set of sequences $\{\gamma_n\}_{n=0}^{\infty}$ of complex numbers which are bounded by one and such that if some term has unit modulus, then all subsequent terms are zero. The numbers $\gamma_n = f_n(0)$, $n \geq 0$, are the *Schur parameters* of $f(z)$. This method of labeling S by numerical sequences is known as the Schur algorithm and is due to Schur [33].

The *Schur problem*, or *Carathéodory-Fejér problem*, is to find conditions for the existence of a function in S whose initial Taylor coefficients are given numbers $c_0, c_1, \dots, c_n$. Schur [33] showed that such a function exists if and only if the lower triangular matrix

$$(1) \qquad \begin{pmatrix} c_0 & & & \\ c_1 & c_0 & 0 & \\ & \ddots & \ddots & \\ c_n & & c_1 & c_0 \end{pmatrix}$$

is bounded by one as an operator on complex Euclidean space, and he determined how all solutions can be found. The method was adapted to Pick-Nevanlinna interpolation by Nevanlinna [29]. This variant of the problem asks to find a function in S which takes given values $w_1, \dots, w_n$ at specified points $z_1, \dots, z_n$ in D.

About the same time, Szegö [34] created his theory of orthogonal polynomials on the unit circle. Let $w(\theta)$ be a nonnegative 2π-periodic measurable function on the real line with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w(\theta)\,d\theta = 1.$$

A unique system of polynomials $\{\phi_n(z)\}_{n=0}^{\infty}$ exists such that

(i) for all $n = 0, 1, 2, \dots$, $\phi_n(z)$ has precise degree n and the coefficient of z^n is real and positive, and

(ii) for all $m, n = 0, 1, 2, \dots$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_m(e^{i\theta})\overline{\phi_n(e^{i\theta})}w(\theta)\,d\theta = \delta_{mn}.$$

Among their properties are limit theorems when the weight has a factorization

$$w(\theta) = |D(e^{i\theta})|^2,$$

where $D(z)$ is an outer function in the Hardy class H^2 on the unit disk and $D(0)$ is positive ("outer" means that the functions $D(z), zD(z), z^2D(z), \dots$ span a dense subspace of the Hardy space—in particular, $D(z)$ has no zeros in the unit disk). For all points in the unit disk,

$$\lim_{n\to\infty} \phi_n^*(z) = 1/D(z),$$

where $\phi_n^*(z) = z^n\bar{\phi}_n(1/\bar{z})$ for all n. The representation of $w(e^{i\theta})$ in terms of $D(z)$ is called a spectral factorization in applications.

The connection between the Schur and Szegö theories is often attributed to Akhiezer [4], but it appears earlier and in greater detail in Geronimus [18, 19]. The connection is based on important recurrence relations which were first given in the 1939 edition of Szegö's book [36] (a version appears also in [35]). The relations were written by Geronimus in terms of the monic polynomials $P_n(z) = \phi_n(z)/\phi_n^*(0)$ in the form

$$P_{n+1}(z) = zP_n(z) - \bar{a}_n P_n^*(z), \qquad n \geq 0,$$

for certain parameters a_n, $n \geq 0$. Since

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} w(\theta)\,d\theta$$

has positive real part in the unit disk and value 1 at the origin, the function

$$f(z) = \frac{1}{z}\frac{g(z)-1}{g(z)+1}$$

belongs to the class S. Geronimus showed that the Schur parameters for $f(z)$ coincide with the numbers a_n in the recurrence formula. In current terminology the numbers $P_{n+1}(0) = -\bar{a}_n$ are called *Szegö parameters*. This relationship and other fundamental identities relating the Schur and Szegö theories remain valid when the orthogonal polynomials are defined relative to a measure instead of a weight function.

An essential generalization of interpolation theory was made by Sarason [31], who saw the Carathéodory-Fejér and Pick-Nevanlinna problems as belonging to the realm of Beurling's theory of invariant subspaces for the shift operator. Let K be a closed invariant subspace of the transformation $f(z) \to [f(z) - f(0)]/z$ on the Hardy space H^2. By Beurling's theorem $K = H^2 \ominus BH^2$ for some inner function B (inner means that multiplication by $B(z)$ is an isometry on the Hardy space—equivalently, the boundary function $B(e^{i\theta})$ has unit modulus a.e. on the unit circle). Let U be the compression of multiplication by z to K, that is,

$$U\colon f(z) \to P_K z f(z),$$

where P_K indicates projection onto K. The operator U is interesting in its own right, and Sarason's generalized interpolation theorem computes its commutant $C(U)$. Every element A of H^∞ induces an operator T_A on K defined by

$$T_A\colon f(z) \to P_K A(z) f(z).$$

The operator T_A only depends on the coset $A + BH^\infty$ in H^∞/BH^∞, and the mapping $A + BH^\infty \to T_A$ is an isometric Banach algebra isomorphism from H^∞/BH^∞ in its quotient norm onto $C(U)$. The space H^∞ is the set of bounded analytic functions on the unit disk in the supremum norm. The infimum which defines the quotient norm is always attained, and the isomorphism is well behaved with respect to the weak operator topology on $C(U)$. Classical solutions to the Carathéodory-Fejér and Pick-Nevanlinna problems are obtained when K is chosen to be the span of the functions $1, z, \dots, z^n$ or $1/(1 - \bar{z}_1 z), \dots, 1/(1 - \bar{z}_n z)$.

The commutant lifting theorem of Sz.-Nagy and Foiaş [37] is another important generalization. Let A be a contraction operator on a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ which satisfies a relation of the form $AT_1 = T_2 A$, where T_1, T_2 are fixed contraction operators on $\mathcal{H}$, $\mathcal{K}$, respectively. If U_1, U_2 are minimal isometric dilations of T_1, T_2 on Hilbert spaces $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{K}}$, the lifting theorem asserts that there exists a contraction operator $\widetilde{A}$ on $\widetilde{\mathcal{H}}$ into $\widetilde{\mathcal{K}}$ such that $A = P_{\mathcal{K}} \widetilde{A}|_{\mathcal{H}}$, where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$. Sarason's theorem is recovered when T_1, T_2 coincide with a compression of the shift operator as described above.

The operator generalizations of interpolation theory at first kept the Schur ideas in the background, but other work dating from about the same time brought them to prominence. One group was centered around V. P. Potapov and was originally relatively little known in the West. This situation changed thanks to private translations by T. Ando of key works of Potapov, Kovalishina, and Katsnel′son [25, 26]. Motivated in part by engineering applications, Potapov and his colleagues derived matricial generalizations of Schur's method including continuous analogues. See Arov [5] for a recent survey and references. Adamjan, Arov, and Kreĭn [1–3] were inspired not only by interpolation theory but also by moment and approximation problems related to Hankel operators and Nehari's problem.

Further development of the commutant lifting problem by Ceauşescu and Foiaş [8] and Arsene, Ceauşescu, and C. Foiaş [6] reconciled the commutant lifting approach with the Schur algorithm. The commutant lifting problem can be reduced to an iterative construction. Each step involves solving an operator equation, and certain free parameters determine all extensions. The parameters are contraction operators and called a *choice sequence*. In a special case corresponding to the Carathéodory-Fejér problem, they coincide with the Schur parameters. In general, they parametrize all solutions of the commutant lifting problem. The monograph by Foiaş and Frazho [17] is an authoritative account which includes applications.

Our account has many omissions, and different viewpoints may be seen in Ball, Gohberg, and Rodman [7], Dym [16], Kreĭn and Nudel′man [27], and other works. Engineering applications have been important for the mathematical development of the subject. See, for example, Helton et al. [21]. Matrix generalizations and applications of the Schur and Szegö theories specifically are treated in a number of places, such as Dewilde and Dym [14]. See also Delsarte and Genin [13], Kailath [23, 24], and other articles in the collection [20].

The book by Bakonyi and Constantinescu is a new source for diverse classical and modern topics. Its main features are:

- accounts of both the Schur and Szegö theories, showing their connections as well as traditional and operator methods,
- a variety of applications, including spectral factorization, matrix orthogonal polynomials, and topics in linear systems and control theory, and
- use of choice sequences as a principal tool.

The book exhibits a point of view favored by the Romanian school of operator theory and how it relates to applications. Most readers will have prior familiarity with at least some of the topics, but in principle the book is accessible to students with experience in the basic methods of operator theory.

Chapter 1 gives an account of the Schur algorithm and its application to the Schur problem. Parallel results are obtained for the *Carathéodory problem*, which is to characterize numbers $s_0/2, s_1, \ldots, s_n$ which are the Taylor coefficients of an analytic function having nonnegative real part on the unit disk. A solution of the latter exists if and only if the matrix

$$\begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ \bar{s}_1 & s_0 & \cdots & s_{n-1} \\ \vdots & \vdots & & \vdots \\ \bar{s}_n & \bar{s}_{n-1} & \cdots & s_0 \end{pmatrix} \tag{2}$$

is nonnegative as an operator on complex Euclidean space. An operator extension of the Schur algorithm is derived. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, let $S(\mathcal{H}_1, \mathcal{H}_2)$ be the set of analytic functions on the unit disk whose values are contraction operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The defect operators of a contraction operator T are defined by $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$; the defect spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are the closures of their ranges. If $F(z)$ is in $S(\mathcal{H}_1, \mathcal{H}_2)$, put $F_0(z) = F(z)$ and $\Gamma_0 = F_0(0)$. If $F_0(z), \ldots, F_n(z)$ and $\Gamma_0, \ldots, \Gamma_n$ have been chosen, $F_{n+1}(z)$ is defined as the unique solution in $S(\mathcal{D}_{\Gamma_n}, \mathcal{D}_{\Gamma_n^*})$ of the equation

$$F_n(z) = \Gamma_n + zD_{\Gamma_n^*} F_{n+1}(z)(I + z\Gamma_n^* F_{n+1}(z))^{-1} D_{\Gamma_n}.$$

The operators $\Gamma_0, \Gamma_1, \ldots$ are called the Schur parameters of $F(z)$. The Schur parameters label the class $S(\mathcal{H}_1, \mathcal{H}_2)$. The construction coincides with the Schur algorithm in the scalar case.

Chapter 2 gives a choice sequence description of contractive matrices of the form (1) and nonnegative matrices of the form (2) in their block operator generalizations. The choice sequence parameters are shown to be identical to the Schur parameters for the coefficient extension problem associated with the lower triangular Toeplitz matrix (1). Connections between the selfadjoint Toeplitz matrix (2) and Naimark dilations and the trigonometric moment problem are also discussed. The results of Chapter 2 draw on work by Constantinescu [9–11] and are central to the point of view of the book.

Spectral factorization is treated in Chapter 3. In the scalar case, the problems are solved by theorems of complex analysis which give the exact multiplicative structure. Noncommutative generalizations of these theorems are not available, however, and there is a long history of alternative methods going back to V. N. Zasuhin, M. G. Kreĭn, N. Wiener, P. R. Masani, H. Helson, and D. Lowdenslager. An approach via choice sequences is presented by Bakonyi and Constantinescu. This yields a theorem of Suciu and Valuşescu [32] which assures the existence of a maximal factorable summand for an operator-valued measure. Particular cases yield results which are obtained in different ways in Sz.-Nagy and Foias [37] and Rosenblum and Rovnyak [30].

Szegö's theory of orthogonal polynomials and its connections with the Schur algorithm are the subject of Chapter 4. In the scalar case this reduces to the classical theory described above. Operator generalizations are derived. Engineering applications are discussed in Chapter 5, which has section titles of realization theory, Darlington synthesis, and stabilizing control. Chapter 6 treats other extension problems such as Nehari's problem and commutant lifting.

Some limitations might be noted. The work is self-contained with respect to the operator methods used, but a price must be paid to follow the sometimes intricate choice sequence arguments. The index, bibliography, and literature notes are thin. The closely related monograph by Dubovoj, Fritzsche, and Kirstein [15] and the companion historical collection [22] have carefully annotated bibliographies and should be consulted to complete the picture.

Interest in operator methods in interpolation theory continues unabated. The reason, perhaps, is that the concepts appear in so many places, such as invariant subspace theory, Kreĭn-space operator theory, and problems of spectral theory in which a prominent role is played by analytic functions which are bounded by one or which have positive real part on a disk or half-plane. To name one example, the theory of Hilbert spaces of entire functions by de Branges [12] is a generalization of the theory of orthogonal polynomials whose connections with Szegö's theory have been explored in a recent paper by Xian-Jin Li [28]. Others would no doubt cite different areas for potential future development. The dialogue between mathematicians and engineers also accounts for much of the activity in the field today, and this should continue for some years hence.

The idea of Bakonyi and Constantinescu to base their book on the classical Schur and Szegö theories is a happy and successful one, and its publication should contribute to the future development of this fascinating subject.

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