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Wavelet theory and its applications, by Randy K. Young. Kluwer Academic Publishers, Boston, Dordrecht, and London, 1993, vix + 223 pp., $67.50. ISBN 0-7923-9271-X

At the end of the twentieth century, which sometimes is called the century of information processing, applied mathematics has become increasingly interested in the areas of signal analysis, high-resolution image processing, adaptive filter theory, and the implementation of neural network models for pattern recognition and associative memory. One of the problems of these rapidly evolving areas of high technology that has been receiving great attention is the efficient encoding and decoding of analog and digital information. Because the capacity of information transmission channels is always limited, a considerable amount of research has been spent on data compression techniques in recent years. Typical applications that benefit from data compression techniques are high-speed modems for computer communications and CD-ROMs scanned by wavelength-tunable laser diodes. High-technology applications like video, high-definition television (HDTV), and visualization in biomedical computing need fast algorithms for time compression and perfect reconstruction of univariate and multivariate signals. The technique of subband coding used in high-performance image processing for data compression purposes combined with the concept of quadrature mirror filter bank for signal reconstruction form important examples of high-tech math. Data compression techniques that are actually available allow the transmission of more than one hundred TV programs through a single low-loss wideband photonic fiber transmission channel. Due to a data-parallel-processing minisupercomputer, the real-time display of holographic images has recently become a reality at the MIT Media Laboratory. Recent progress in computerized hoxel (holographic element) technology suggests that holographic TV for the display of three-dimensional images might be available within the next ten years. Holographic TV will provide even more impressive images than HDTV.

What is more, the growing interest in the areas of signal analysis, high-resolution image processing, adaptive transversal filter theory, and the implementation of synchronized neural network models for pattern recognition and associative memory revealed the mathematical fundamentals of a more advanced information theory called quantum information theory. It embeds the classical electrodynamics into a macroscopic quantum field theory of bosons and interprets the coherent wavelet transform as a relative quantum entropy density function. The quantum entropy interpretation is of particular importance for synchronized neural network models. Although the range of this highly promising field of research is presently unknown, powerful implementations of macroscopic quantum field phenomena and their signal response are already available. Macroscopic quantum field phenomena occur in the quantum Hall effect in metal-oxide-semiconductor field effect transistors and heterostructures at low-temperature and high-magnetic fields, in the power spectra of ultrasonic cavitation oscillations in coherently driven liquids [7], as well as in many fields of remote sensing where the phase of the coherent wavelet may be measured as well as its magnitude. Such filtering measurements may be viewed to contain both a semantic content represented by the phase and a level of confidence in that content represented by the magnitude of the coherent wavelet. Examples of macroscopic quantum field phenomena and signal analysis of their response occur in clinical and biomedical imaging, underwater acoustic imaging, radar satellite imaging, geophysical tomography, and multichannel seismic imaging. It passes by various names, including quantum holography and synthetic aperture radar (SAR) imaging [6]. Besides satellite SAR, the most sophisticated and spectacular applications of phase coherent wavelets providing invaluable support to radiological diagnosis are spatially localized nuclear magnetic resonance imaging (MRI), magnetic resonance mammography (MRM), and magnetic resonance angiography (MRA), which revolutionized noninvasive clinical imaging and visualization in biomedical computing by detecting and processing coherent spin wavelets or magnons. Exactly as the word magnon describes the wavelets of the macroscopic system of electron spins coupled together by exchange interactions, photons represent by duality the particle aspect of electromagnetic wavelets in vacuum. The photonic counterpart of MRI holograms is formed by off-axis laser transmission holograms possessing horizontal parallax only. Due to the capability to determine the distinct chemical signatures of different tissue types and multifocal lesions, MRI scanners form a real breakthrough in radiological diagnosis, far surpassing X-ray transmission computed tomography (CT) scanners. Actually, clinical MRI owes much to the development of CT. Although even the pioneers in high-resolution nuclear magnetic resonance spectroscopy (MRS) never believed diagnostic MRI would work—like Rutherford, who said that anyone who believed nuclear radioactivity would be useful "is talking moonshine"—MRI has become one of the main radiological tools available at all major diagnostic centers all over the world. It admits ray-tracing fan postprocessors like massively parallel quantum holographic neurocomputers which are useful for three-dimensional visualization of stacks of individual tomographic planar slices. The three-dimensional display of the human nervous system has recently become a reality at the University of Washington in Seattle. The visualization in biomedical computing supports physicians by facilitating radiological diagnosis and therapy control, both of which depend mainly on noninvasive visual information.

Wavelet theory is nowadays a very active field of approximation theory with a wide impact on signal analysis, high-performance imaging applications, and adaptive transversal filter theory. It is concerned with the modeling of univariate and multivariate signals with a set of specific signals. The specific signals are just the wavelets. Families of wavelets are used to approximate a given signal (with respect to the L^2 norm, say), and each element in the wavelet set is constructed from the same original window, the mother wavelet. In the affine wavelet set, the elements are time-scaled (dilated or compressed) and time-translated (shifted) replicas of the mother wavelet. The time-scaling operation includes normalization so that the intensity of the original mother wavelet is preserved. Many desirable advantages exist for using spline functions to derive mother wavelets. In the case of coherent wavelets the scale parameter is replaced by another synchronization parameter, the time-dependent differential phase factor. In this context "differential" refers to phase differences between the object and reference wavelets being processed and not to the phase of either wavelet alone. It is the differential-phase variable which represents one quantum of intelligent association within synchronized neural network models, and multichannel magnetoencephalography proves that coherent oscillations within the 40 Hz frequency band (40 Hz EEG) represent the reference wavelets of the synchronized bursts generated by the central nervous system of mammalians during sensory encoding, cognitive processing and motor behaviour. The symplectically invariant Weyl symbol in the sense of pseudodifferential operators [3] determines by the differential-phase variable and the parallax bundle, i.e., the principal fiber bundle of symplectic frames over the individual hologram plane, the quantum holograms of coherent photonic signal processing and the interference patterns of coherent wavelets in synchronized neural network models. Similarly, the Weyl symbol determines by the differential-phase variable and by the controlled magnetic field linear gradient bundle over the selective planar spin slice the MRI holograms which are formed by the holo-lines of the learning matrices [1] of coherent spin wavelets in spatially localized MRI. In the coherent case, which is as useful as affine wavelets for image compression purposes, the hoxel-based scaling or warping procedure as well as the relations to fractals and macroscopic chaotic dynamical systems of photons, acoustic phonons, magnons, and so on is based on the stroboscopic projection of the discrete Heisenberg subgroup. The phase coherent wavelets are more delicate than the affine wavelets but afford the greatest spatial sampling efficiency.

The implementations of synchronized neural network models that are based on coherent wavelets as the commerically available holographic neural technology and the analog complementary metal-oxide-silicon very large-scale integrated early vision chip of high-wiring complexity [4] are closer to physiological neural networks and therefore different from the ADALINE and MADALINE perceptrons [1] of adaptive transversal filter theory and robotic path planning.

From the mathematical point of view, the generation of the wavelets from a mother wavelet can be looked at as a G-orbit where G denotes the affine ($'\alpha t + \beta'$) solvable Lie group of real matrices

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}, \qquad (\alpha > 0)$$

in the case of the affine wavelets [2] with average-width parameter α and synchronization parameter β, and the real Heisenberg nilpotent Lie group of matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix}$$

in the case of coherent wavelets [5] with synchronization parameters due to the signal response of macroscopic quantum field phenomena. In this context the Lie group G represents the basic symmetries to analyze and synthesize the wavelets which form the input and output signals. To determine the unitary dual of G which encodes important properties of the wavelets, notice that the connected affine group of the real line $\mathbf{R}$ admits only two nontrivial, noninteractive, coadjoint orbits represented by complementary open half-planes. The three-dimensional real Heisenberg group, however, implements symmetries at the quantum level and admits infinitely many nontrivial, noninteractive, coadjoint orbits. These are all transversal to the center line and planar, uniquely determined by the central character, and endowed with a natural symplectic structure due to the fundamental covariance identity of the unitary projective oscillator representation of the symplectic group $\mathrm{Sp}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})$. Thus the generic nontrivial coadjoint orbit of the real Heisenberg group forms a cross-section in the tangent bundle $T\mathbf{S}_2$ of the compact unit sphere $\mathbf{S}_2$ of the real Euclidean vector space $\mathbf{R}^3$. There exists a noncanonical bijection between the set of all symplectic frames at a point of the base manifold $\mathbf{S}_2$ and the linear structure group $\mathrm{Sp}(2, \mathbf{R})$ of the principal fiber bundle over $\mathbf{S}_2$.

From the physical point of view, the wavelets represent G-coherent states, and the natural dual reductive pair $(\mathrm{Sp}(2, \mathbf{R}), \mathrm{O}(2, \mathbf{R}))$ inside the symplectic subgroup $\mathrm{Sp}(4, \mathbf{R})$ of the special linear group $\mathrm{SL}(4, \mathbf{R})$ determines by the decomposition of the unitary projective oscillator representation of $\mathrm{Sp}(4, \mathbf{R})$ into discrete series representations the controlled magnetic field linear gradient bundle. Notice that the automorphism group $\mathrm{SL}(2, \mathbf{C}) = \mathrm{Sp}(4, \mathbf{R})$ of the complexified Heisenberg group letting its center pointwise fixed admits $\mathrm{SU}(2, \mathbf{C})$ as a maximal compact subgroup. The corresponding maximal subgroup $\mathrm{SO}(2, \mathbf{R})$ of $\mathrm{Sp}(2, \mathbf{R}) = \mathrm{SL}(2, \mathbf{R})$ generates the holographic lattice from the stroboscopic projection of the discrete Heisenberg subgroup on the individual hologram plane. The rows of the holographic lattice represent the phase-encoding steps and hence the spatial resolution of the quantum hologram.

The natural action of $\mathrm{SU}(2, \mathbf{C})$ on $\mathbf{R}^4$ spanned by the Pauli spin matrices induces the adjoint action of $\mathrm{SU}(2, \mathbf{C})$ on its Lie algebra which is a rotation of $\mathbf{R}^3$ parametrized by the Euler angles. The transitive group action of $\mathrm{SU}(2, \mathbf{C})$ on $\mathbf{S}_3$ allows one to identify the image $\mathbf{S}_2$ of $\mathbf{S}_3$ under the Hopf mapping with a compact $\mathrm{SU}(2, \mathbf{C})$-homogeneous manifold. The Hopf projector consists of keeping the declination and azimuth angles for tomographic coronal and sagittal planar spin slice imaging and forgetting about the third Euler angle. To optimize the spherical harmonic homogeneity of the static magnetic field, which is particularly crucial in high-resolution nuclear MRS, spatially localized MRI, MRM, and MRA, the magnetic field corrections determine the geometry of the toroidal and saddle electrical shim and linear gradient surface coils inside the bore of the superconducting magnet of the diagnostic MRI scanner. Specifically, Maxwell pairs, Golay coils, quadrupole sets, and so on are designed in terms of the group $\mathrm{O}(2, \mathbf{R})$ which is the union of $\mathrm{SO}(2, \mathbf{R})$ and a plane reflection copy $\mathrm{SO}^{\vee}(2, \mathbf{R})$, and zonal and tesseral surface spherical harmonics or associated Legendre functions. A part of the shimming and controlled magnetic field linear gradient strategy for the macroscopic quantum field phenomenon of superconductivity, harmonic analysis of the natural dual reductive pair $(\mathrm{SL}(2, \mathbf{R}), \mathrm{O}(2, \mathbf{R}))$ provides the singular value decomposition of the Radon transform of CT in terms of tensor products of Laguerre and Hermite functions, respectively, with surface spherical harmonics. Thus the natural duality between symplectic and orthogonal groups links the theory of automorphic forms to the generation of coherent wavelets in electrical engineering and underpins the coherent wavelet transform with a deep string-like theory. Because Hermite functions form the weighted matching polynomials of complete bichromatic graphs, the connection of quantum holography to the layered architecture of synchronized neural network models [6] becomes evident.

From the electrical engineering point of view, the wavelets give rise to the wideband and narrowband ambiguity functions, respectively. The wideband width capabilities of photonic fiber channel networks are attractive features of photonic telecommunication systems, whereas narrowband width frequencies are used in spatially localized MRI for selective planar spin slice excitation purposes. It is this variety of different approaches which makes wavelet theory such an exciting field of multidisciplinary research.

The theory of affine and coherent wavelets is still at a relatively early stage of evolution, and new results and applications are developing rapidly [8]. The book under review has its origin in a Ph.D. thesis submitted by the author to the Department of Electrical Engineering of the Pennsylvania State University at University Park, PA, in 1991. Its purpose is "to present wavelet theory so that it is accessible to a broader audience than the current readers of research." For image processing and filter theory, the book centers around the multiresolution, the orthogonal or biorthogonal affine wavelets, and the wideband matched filter concept. These kinds of wavelets and their desirable properties are presented and discussed in a nontechnical manner along with diagrams. Unfortunately, the pyramidal structure which is built in multiresolution wavelet transforms is not described in terms of digital prefilters, sampling, and digital postfilters of interpolating polynomial spline functions. Therefore, an efficient implementation of multiresolution wavelet algorithms does not become obvious for the reader.

The irreducible unitary linear representations of the affine group and the Heisenberg group admit many different concrete realizations which can be organized into families of wavelets. This circumstance seems to be at the heart of the theory of wavelets. In the book under review the extremely powerful group representational approach to the theory of wavelets and their applications is completely neglected. In the case of coherent wavelets the author therefore misses the important point of tomographic planar slice modeling for phase encoding in the macroscopic quantum field theory due to the restricted repertory of tools that are available for his mathematical exposition. Can a book that requires only college-senior-level mathematics give a realistic account of a comprehensive theory whose mathematical origin can be traced back to the work by Alberto P. Calderón in the affine case and to the work by Erwin Schrödinger and Hermann Weyl in the coherent case? The reviewer's answer is a clear "No, it cannot." The title of the book is not well chosen, because the book lacks a genuine mathematical theory and includes only some specific applications. It can provide, however, the motivation to mathematicians not to learn just another theory but to understand how powerful good mathematics can actually be for applications in the areas of signal analysis, high-performance image processing, and filter theory. In view of the aforementioned powerful applications of coherent wavelets, the bias toward affine wavelets and, hence, the restriction to the application of affine wavelets to wideband telecommunication purposes is not justified. Nevertheless, the book can be recommended as a first survey of high-tech math, as a source of useful information, and as the starting point of a more advanced and deeper study of the extremely flexible technique of affine and coherent wavelets. From an advanced study of the G-coherent states approach to wavelets, the reader will recognize that macroscopic quantum field phenomena and signal analysis of their response are actually the keys to high technology and neural networks. Wavelets form an appropriate tool to leave the classical theory behind.

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Dimension and extensions, by J. M. Aarts and T. Nishiura. North-Holland Math. Library, Amsterdam, London, New York, and Tokyo, 1993, xii + 331 pp., $106.50. ISBN 0 444 89740 2

Johann de Groot would have loved this book! In his 1942 doctoral dissertation he considered the problem of compactifying a (separable metrizable) space X by adding a metrizable remainder of minimal dimension. This latter dimension is called the *compactness deficiency*, $\operatorname{def} X$, of the space X. He formulated a conjecture on how to characterize this number internally that captured the imagination of scores of mathematicians. In 1980 Roman Pol disproved this conjecture but only after a new branch of dimension theory had been developed and had taken on a life of its own. This excellently written, exciting book is a portrait of a living and dynamic area as well as a monument to de Groot built by