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# BOOK REVIEW

*Dimension and extensions*, by J. M. Aarts and T. Nishiura. North-Holland Math. Library, Amsterdam, London, New York, and Tokyo, 1993, xii + 331 pp., \$106.50. ISBN 0 444 89740 2

Johann de Groot would have loved this book! In his 1942 doctoral dissertation he considered the problem of compactifying a (separable metrizable) space $X$ by adding a metrizable remainder of minimal dimension. This latter dimension is called the *compactness deficiency*, $\operatorname{def} X$, of the space $X$. He formulated a conjecture on how to characterize this number internally that captured the imagination of scores of mathematicians. In 1980 Roman Pol disproved this conjecture but only after a new branch of dimension theory had been developed and had taken on a life of its own. This excellently written, exciting book is a portrait of a living and dynamic area as well as a monument to de Groot built by his former student, Jan Aarts, and his able coauthor, Togo Nishiura. It should be required reading for anyone interested in dimension theory. Regrettably, its high price will prevent it from being on as many bookshelves as it should.

The first chapter sets the stage for the rest of the drama of the book. It concentrates on spaces that are separable and metrizable where the small inductive, the large inductive, and the classical Lebesgue covering dimensions coincide. For such a space $X$, define $\operatorname{cmp} X$ inductively as follows: (i) $\operatorname{cmp} X = -1$ if $X$ is compact; (ii) for any nonnegative integer $n$, $\operatorname{cmp} \leq n$ if for each point $p$ and each closed set $G$ not containing $p$ there is a partition $S$ between $p$ and $G$ with $\operatorname{cmp} S \leq n-1$; (iii) $\operatorname{cmp} X = n$ if $\operatorname{cmp} X \leq n$ and $\operatorname{cmp} X \not\leq n-1$; and (iv) $\operatorname{cmp} X = \infty$ if the inequality $\operatorname{cmp} X \leq n$ fails for each $n \geq 0$. (This would be the definition of $\operatorname{ind} X$ if "$X$ is compact" in (i) were replaced by "$X = \varnothing$".) In his thesis de Groot showed that $\operatorname{def} X = \operatorname{cmp} X$ (for $X$ separable and metrizable) if $n = -1$ or 0, and he conjectured that this equality held for every positive integer. The corresponding analog of the large inductive dimension is denoted by Cmp. Many other dimension functions were introduced in an attempt to split the de Groot problem into more tractable pieces, and it was less than obvious that one was more pertinent than the other. In 1960 Sklyarenko introduced an integer-valued function that associates with separable metrizable $X$ an integer the authors denote by $\operatorname{Skl} X$ as follows: $\operatorname{Skl} X \leq n$ if $X$ has a base $\mathcal{B} = \{B_i \colon i = 1, 2, \dots\}$ such that for any $n+1$ different indices $i_0, \dots, i_n$, the intersection of the boundaries of the $U_{i_j}$ for $0 \leq j \leq n$ is compact.

Inductive definitions are not easy to work with, and it took until 1982 for Pol to produce an example of a separable metrizable space $X$ with $\operatorname{cmp} X = 1$ and

$\operatorname{Cmp} X = \operatorname{def} X = 2$. Later, Kimura and then Hart showed that the gap between $\operatorname{def} X$ and $\operatorname{cmp} X$ could be made arbitrarily large. In 1988 Kimura was able to show that $\operatorname{def} X = \operatorname{Skl} X$. While this may appear to put the finishing touch to the de Groot problem, it is far from the end of the story—even for the separable metrizable case. Compactifying is not the only way to extend a space. For example, the corresponding questions for completions are investigated. Covering dimension is also generalized. The many unsolved problems in this chapter demonstrate how alive this subject remains even in the separable metrizable case.

While the material described above occupies less than a quarter of the book, it sets the tone for the rest of it. Once one lifts the restriction to separable metrizable spaces, the classical dimension functions part company, as do their generalizations in the spirit of de Groot. The authors consider generalizations of dimensions modulo classes of topological spaces. A unified approach to the various compactifications that arise in this kind of dimension theory is developed. A lot of new material is presented, and unsolved problems abound. To give the rest of the book the attention it deserves would require an encyclopedic review and would embarrass the reviewer if he failed to convey the excitement one gets by reading this volume.

Many topics considered are motivated and placed in proper historical perspective. Generality appears naturally, not just for its own sake. While a prior knowledge of classical dimension theory is helpful, the book is largely self-contained except for references to older books for some proofs of theorems. The exposition is masterful, and it is clear that the authors are more interested in communication than in displaying their (considerable) erudition.

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