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BOOK REVIEW

Multidimensional integral transformation, by Ya. A. Brychkov, H. J. Glaeske, A. P. Prudnikov, and Vu Kim Tuan. Gordon & Breach Science Publishers, Philadelphia, PA, 1992, xiii + 386 pp., $76.00. ISBN 2-88124-839-X

The concept of an integral transformation is familiar. If a function f is defined on a space Ω and the function $\tilde{f}$ is defined by an equation of the form

$$\tilde{f}(y) = \mathcal{T}[f(x); x \to y] = \int_\Omega f(x)K(x,y)\,dx \tag{1}$$

where the function K is defined on $\Omega \times \Omega^*$, the usual practice is to call the operator $\mathcal{T}\colon \Omega \to \Omega^*$ an *integral transformation* and the function $\tilde{f}$ an *integral transform*, though in this volume there is a certain confusion and the terms are regarded as interchangeable.

In Chapter IX of his treatise [1], in discussing the diffusion of heat in an infinite solid, Fourier introduced—but hardly could be said to have proved!—theorems concerning double integrals which were later interpreted as theorems involving *Fourier transforms.* Similarly what we now call *Laplace transforms*, which in the notation of (1) have Ω equal to the positive real line and $K(x,y) = e^{-xy}$, may be seen to have origins in Chapter 1 of Part 2, Volume 1 of Laplace's treatise on probability [2].

The first recognized use of an integral transform appears to have been due to Riemann [3]. In considering the Dirichlet series

$$\Phi(s) = \sum_{n=1}^{\infty} a_n/n^s,$$

Riemann introduced the function

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-nx}.$$

We then have

$$\Phi(s) = f^*(s)/\Gamma(s),$$

where

$$f^*(s) = \mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}\,dx$$

is what is now called the *Mellin transform* of f.

In the early days of the theory of integral transforms Ω was usually either the real line or the positive real line, and the functions f were almost always continuous functions of the class $L^1(\Omega)$. When we examine the limitations involved in applying these methods to the theory of partial differential equations, we soon appreciate the usefulness of working on sets of distributions much more extensive than the sets of continuous functions envisaged in the classical theory. Also, Ω is taken to be $\mathbb{R}^n$ or some subsets of it; this gives the description *multidimensional* to the transforms considered here.

In their preface the authors state, "The aim of this book is to provide an introduction to multidimensional transformations. It is a cross between a textbook for students of mathematics, physics, and technology and a monograph on this subject. The main topic under consideration is the operational calculus of the most important integral transformations and some if its applications. Here we investigate transforms in spaces of functions as well as transforms in spaces of distributions, though the latter are only briefly discussed."

In this they have been only partially successful, perhaps because of trying to reach too wide an audience.

After a preliminary chapter, Chapter 0, in which in eight pages they discuss numbers, Euclidean and unitary spaces, multi-indices, spaces of functions, and some theorems for L^1-functions, they begin with a chapter on the Fourier transformation. They define the Fourier transform $\hat{f}(\mathbf{y})$ of a function $f(\mathbf{x}) \in L^1(\mathbb{R}^n)$ by the equation

$$\hat{f}_n(\mathbf{x}) = \mathbf{F}_n[f\,](\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\mathbf{xy}} f(\mathbf{y})\,d\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{2}$$

adding the quite unnecessary remark (since $f \in L^1(\mathbb{R}^n)$) "provided that the integral exists". They add that sometimes other notations are more convenient; actually the one they have chosen is the *most inconvenient*, since its inversion theorem takes the form $\mathbf{F}_n^{-1}[\hat{f}\,](\mathbf{y}) = (2\pi)^{-n}\mathbf{F}_n[\hat{f}\,](-\mathbf{y})$, whereas in the two alternatives they mention the factor $(2\pi)^{-n}$ is replaced by unity.

There is a similar difficulty with their definition of the *Hankel transform*, which they introduce in this first chapter on Fourier transforms. In working with Hankel transforms, pure mathematicians use the transform

$$\langle_\nu[f\,](r) = \int_0^\infty J_\nu(2\sqrt{r\rho}) f(\rho)\,d\rho,$$

whereas applied mathematicians use the definition

$$\mathcal{H}_\nu[f(r);\xi] = \int_0^\infty r f(r) J_\nu(r\xi)\,dr,$$

each having the property that the operator defining it is self-reciprocal. The definition adopted by the present authors,

$$\overline{f}_\nu(r) = \mathbf{H}_\nu[f\,](r) = \int_0^\infty \rho^{1/2} f(\rho) J_\nu(\rho r)\,d\rho,$$

does not have this property—its inversion theorem states that $\mathbf{H}_\nu^{-1}[\overline{f}\,](\rho) = \rho^{1/2}\mathbf{H}_\nu[r^{1/2}f(r)](\rho)$. However, it should be pointed out that Erdélyi et al. [4] and Oberhettinger [5] use the self-reciprocal operator h_ν defined by the equation

$$h_\nu[f\,](r) = \int_0^\infty (\rho r)^{1/2} f(\rho) J_\nu(\rho r)\, d\rho.$$

The extensive tables given by these authors for this transform allow us to determine $\mathbf{H}_\nu$-transforms by use of the formula $\mathbf{H}_\nu[f\,](r) = r^{1/2}h_\nu[f\,](r)$.

If we use the operators $\mathcal{F}_n = (2\pi)^{-n/2}\mathbf{F}_\kappa$ and $\mathcal{H}_n$, the somewhat clumsy formula (1.24′) assumes the simpler form: If $r = |\mathbf{x}|$, $\rho = |\mathbf{y}|$, then

$$\mathcal{F}_n[f(r)](\mathbf{y}) = \rho^{-\nu}\mathcal{H}_\nu[r^\nu f(r);\rho], \quad \text{with } \nu = \tfrac{1}{2}n - 1.$$

This shows how the present authors' choice of notation leads to formulae which are more complicated than is usual but is not, of course, a serious fault, as long as anyone referring casually to the book makes sure of the precise definitions the authors are using. A more serious trap for the reader is that initially the authors adopt the definitions

$$\mathbf{x}\cdot\mathbf{y} = (x_1y_1,\ldots,x_ny_n) \in \mathbb{R}^n \quad \mathbf{xy} = x_1y_1 + \cdots + x_ny_n \in \mathbb{R}, \qquad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n,$$

but in some later sections of the book interchange them in some basic definitions.

Chapter I deals with the multidimensional Fourier transform both in its classical sense and in the sense of Schwartz's theory of distributions. It begins (in §1.1) with the definition and the basic properties of the Fourier transform of L_1-functions and ends the first section by stating that one "can consider the n-dimensional transform as a *product* of one-dimensional Fourier transforms which are very well tabulated." This, of course, means multidimensional transforms need not be tabulated; however, the same can be said of multidimensional Laplace transforms, but that does not prevent the authors' printing 76 pages of two-dimensional Laplace transforms! This opening section is succeeded by subsections on "Operational Properties", "Fourier Transforms of Radially Symmetric Functions", "Inversion of the Fourier Transform"—all of which are clearly written and easily accessible to the whole audience the authors have in mind—students of mathematics, physics, and technology.

The second section (§1.2) treats the Plancherel theory of the Fourier transform of L_2-functions. This section is much shorter, since it is devoted to the *basic* theory of such transforms, and is likely to be of interest only to students of pure mathematics. The only surprising thing about it is that not even a passing reference is given to Norbert Wiener's book [6].

The next section (§1.3) is not likely to appeal much to students of technology unless they are simultaneously attending graduate courses in Schwartz's theory of distributions. It begins with a subsection of five pages entitled "Spaces of Test Functions and Distributions", which is too brief to act as more than a reference to readers already familiar with the basis of Schwartz's theory. This is followed by subsections on Fourier transforms of rapidly decreasing functions, of tempered distributions, and of entire functions.

The remaining section of this chapter deals with applications of the multidimensional Fourier transform to the solution of partial differential equations; the authors state that, "The main domain of applications of the multidimensional Fourier transform is the use for *the solution of boundary value problems for partial differential equations with constant coefficients*." This is undoubtedly true, but in the applications they consider the problems are mostly **initial value problems** for $\mathbb{R}^n$, not boundary value problems. The first subsection (§1.4.1) discusses the second-order partial differential equation $P(\mathbf{D})u(\mathbf{x}') = f(\mathbf{x}')$, $\mathbf{x}' = (t, \mathbf{x})$ with $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$; here $P(\mathbf{D})$ is a linear differential operator of order 2 in $\{\partial/\partial t, \partial/\partial x_1, \ldots, \partial/\partial x_n\}$ with constant coefficients. It is shown how the application of the operator $\mathbf{F}_n$ reduces an initial value problem for a partial differential equation to one for an ordinary differential equation in t. Subsequent subsections consider the homogeneous heat equation in $\mathbb{R}^n$, the fundamental solution of the Schrödinger operator, and the homogeneous wave equation. It is only in the later subsections and then for the half-space $\mathbb{R}_+ \times \mathbb{R}^n$ that they consider boundary value problems; denoting the coordinates in $\mathbb{R}^{n+1}$ by $(x_0, x_1, \ldots, x_n) = (x_0, \mathbf{x})$, they consider Laplace's equation and Helmholtz's equation in the half-space $x_0 > 0$, in each case satisfying the boundary condition $u(0, \mathbf{x}) = u_0(\mathbf{x})$, the fundamental solution of Poisson's equation, and finally the solution of the biharmonic equation in the half-space $x_0 > 0$ satisfying the boundary conditions $u(0, \mathbf{x}) = u_0(\mathbf{x})$, $\partial u(0, x)/\partial x = u_1(\mathbf{x})$.

There is nothing particularly novel in this section, but it is a good comprehensive review of basic work on partial differential equations and solutions being derived in both the L_1 theory and in the theory of distributions.

Chapter II is given over to the multidimensional Laplace transformation. The authors justify the length of this—at 112 pages the longest in the book—by the importance they attach to the use of Laplace transforms in applications. This is a personal view: it is certainly true that the one-dimensional Laplace transform is more widely used than the corresponding Fourier transform, especially in problems reducing to *ordinary* differential equations. When we turn to *partial* differential equations, the situation would appear to be reversed; and recent work in linear elastostatics, based on the system of Navier equations of equilibrium, makes extensive use of two-dimensional Fourier transforms.

In the first section (§2.1) the foundations of the subject are described: the definition and basic properties; the domain of convergence; uniqueness theorems; and the calculation of the Laplace transform of a function $f(\mathbf{t})$ which can be developed into a series, the Laplace transform of each of whose terms is known. The next section (§2.2) is given over to the development of the operational calculus based on the n-dimensional Laplace transform. This is followed by a section on the inversion theorem; this not only includes the extension to n variables of the one-dimensional complex inversion formula but also inversion formulas of limit type generalising those in one dimension due to Post and Widder. In §2.3.3 and §2.3.4 there is advice on the practical inversion of transforms by combining the use of partial fractions with that of tables and by developing the transform in series, the inverse transform of each of whose terms is known.

Then there comes a subsection (§2.4) on "Convolution Theorems", the contents of which not only contain an account of the properties of an obvious extension to the multidimensional case of the one-dimensional convolution but also those of a generalised convolution. This section ends with the introduction of an extension to n dimensions of the Riemann-Liouville fractional integral and its basic property under the multidimensional Laplace transform.

Abel and Tauberian theorems for multidimensional Laplace transforms are discussed in §2.5, and this is followed by §2.6 entitled "General Relations of Operational Calculus", in which are treated methods of calculating two-dimensional Laplace transforms of functions, like $f(x+y)$, $f(|x-y|)$, and $f(xy)$, in terms of the (one-dimensional) Laplace transform of the function f, and the inverse problem of determining the function $f(x,y)$ when the two-dimensional Laplace transform $F(p,q)$ is known. In §2.6.4 the authors treat a class of problems, the solution of which is required in nonlinear systems analysis. This involves the investigation of an *associated transform*, which is defined as follows: If the function $f(t_1, t_2, \ldots, t_n)$ is given, we define $f_0(t)$ to be $f(t, t, \ldots, t)$ and take the one-dimensional Laplace transform, $F_0(p)$, of this function to be the *associated transform* of the Laplace transform, $F(p_1, p_2, \ldots, p_n)$, of the function f. The associated transform has some interesting properties, which are outlined here.

In §2.7 there are some remarks on the bilateral Laplace transform defined for functions not necessarily vanishing on $\mathbb{R}^n \backslash \mathbb{R}^n_+$ (a nomenclature derived from the one-dimensional case) and on their connection with the Fourier transform.

In §2.8 we have a thorough treatment of the Laplace transform of distributions, with subsections on "Definition and Basic Properties", "Operational Rules", and "The Bilateral Laplace Transform".

Chapter 2 ends with a 26-page section on applications. It begins (in §2.9.1) with a subsection on the evaluation of integrals; this depends largely on the observation that if $F(p,q)$ is the two-dimensional Laplace transform of the function $f(x,y)$, then the integral $\int_0^x f(x-t,t)\,dt$ is the one-dimensional inverse Laplace transform of $F(p,p)$. Four examples are given of how the double transforms may be used in the calculation of integrals, although in one of them—Example 3 on p. 167—this seems an unnecessarily complicated way of evaluating, since it can be written as the difference of two standard (and elementary) integrals.

The next subsection (§2.9.2.) deals with the calculation by the use of two-dimensional Laplace transforms of sums involving some standard special functions. There is no general method of doing this, but the method is illustrated by the consideration of special cases of series involving Laguerre polynomials and Whittaker functions. This is followed by investigations of Laguerre and Hermite polynomials of n variables and of hyper-Bessel functions.

The third subsection deals with the application of the two-dimensional Laplace transform to the solution of linear partial differential equations in two independent variables. The authors concede that in equations in more independent variables it is common to use the Laplace transform with respect to some of the variables and another transform with respect to the remaining ones; but since they do not wish to present such "mixed" methods, they confine themselves to problems in two variables, with the Laplace transform applied to both of them. To illustrate the method in the case of first-order partial differential equations, they consider the boundary value problem $u_x + u_y = f(x,y)$, $(x,y) \in \mathbb{R}$, $u(x,0) = a(x)$, $u(0,y) = b(y)$, $a(0) = b(0)$ and then the corresponding problem when the left-hand side of the differential equation is replaced by $u_x - u_y$. In turning to second-order equations, the authors discuss boundary value problems in $\mathbb{R}^2_+$ for the one-dimensional wave equation, the one-dimensional heat equation, and the two-dimensional Poisson equation.

In the final subsection of this chapter there is a treatment of the so-called Volterra series which was recently featured in a theory of the analysis of nonlinear systems analysis. This is a very helpful account of a topic appearing for the first time in book form.

Chapter 3 deals with the multidimensional Mellin transform in (weighted) L_1 and L_2 spaces and in spaces of distributions. This chapter is likely to be of interest to students of *pure* mathematics, since there does not appear to be in the literature any applications—even to other branches of pure mathematics!—of even the two-dimensional Mellin transform. There certainly is not an account of any applications here.

The final chapter, Chapter 4, entitled "Other Integral Transformations", exhibits the same feature of being of interest only in itself and not in the hope of any applications. In the first section there is introduced the *Mellin convolution type transform* which is defined by the equation

$$g(\mathbf{x}) = \mathbf{K}[f](\mathbf{x}) = \int_{\mathbb{R}^n_+} k(\mathbf{x},\mathbf{y}) f(\mathbf{y})\,d\mathbf{y}, \tag{3}$$

which, if we write $k(\mathbf{x},\mathbf{y}) = \mathbf{y}^{-1} g(\mathbf{x}\mathbf{y}^{-1})$, reduces to the convolution for the multidimensional Mellin transform. This form of integral transform has the advantage that the definition of several transforms can, with a change of variables, be brought to the form (3). The main subsections (§§4.1.2–4.1.4) describe the spaces which are ideally adapted for the study of a large class of transformations of the Mellin convolution type.

In the next section we obtain another generalisation of the one-dimensional Laplace transform by taking $k(\mathbf{x},\mathbf{y}) = \exp[-\max(x_1y_1,\ldots,x_ny_n)]$ in equation (3). The basic properties of such modified Laplace transforms are given in §4.2.1 and §4.2.4; the complex inversion formula for these transforms is derived in §4.2.2 and the real inversion formula in §4.2.3. A second modified Laplace transform is introduced in the final subsection (§4.2.5), where it is defined by equation (3) with $k(\mathbf{x},\mathbf{y}) = \exp[-\min(x_1y_1,\ldots,x_ny_n)]$.

The next section deals with another multidimensional transform of generality—what the authors introduce as the **H**-transform. (It should be observed that this is not a multidimensional extension of the **H**-transform tabulated in Chapter XI of [4].) The definition given here is

$$\mathbf{H}[f(\mathbf{t});\mathbf{x}] = (2\pi i)^{-n} \int_{(\mathbf{1}/2)} H^*(\mathbf{s}) f^*(\mathbf{s}) \mathbf{x}^{-\mathbf{s}}\,d\mathbf{s}$$

where $f^*(\mathbf{s})$ is the n-dimensional Mellin transform of $f(\mathbf{t})$

$$H^*(\mathbf{s}) = \frac{\prod_{j=1}^{A} \Gamma(a^j + \alpha^j s_j)}{\prod_{j=1}^{B} \Gamma(b^j + \beta^j s_j)},$$

where $a^j \in \mathbb{C}$, $\alpha^j \in \mathbb{R}$, $j = 1,2,\ldots,A$, $b^j \in \mathbb{C}$, $\beta^j \in \mathbb{R}$, $j = 1,2,\ldots,B$, and $(\mathbf{1}/2)$ denotes the family of contours $\{s_j \in \mathbb{C} : s_j = \frac{1}{2} + i\tau_j,\ j = 1,2,\ldots,n\}$, the parameters a^j, b^j, α^j, and β^j being chosen in such a way that none of the zeros lies on one of the contours of the set $(\mathbf{1}/2)$. Subsection 4.3.1 deals with the basic properties (including the inversion theorem), and §4.3.2 with the factorization of the **H**-transform. In §4.3.3 the special case in which $H^*(\mathbf{s}) = \Gamma(\mathbf{s})\Gamma(\mathbf{1}-\mathbf{s})$ is considered; this leads to multidimensional Stieltjes transform, for which two forms of inversion formula are derived. In §4.3.4 the case in which $H^*(\mathbf{s}) = 2^{-1/2n+\mathbf{s}\mathbf{1}}\Gamma[\frac{1}{2}(\mathbf{s}+\boldsymbol{\nu}) +$

$\frac{1}{4}\mathbf{i}]/\Gamma[\frac{1}{2}(\mathbf{s}+\boldsymbol{\nu})+\frac{3}{4}\mathbf{i}]$, the corresponding multidimensional transform is one in which the kernel is a product of Bessel functions of the first kind—hence it is called a multidimensional Hankel transform. About the only interesting property possessed by this transform is that it is self-reciprocal, a result which is established in Theorem 4.23 on p. 241. The next subsection deals with the generalisation to n dimensions of the Riemann-Liouville fractional integral and of what the authors call the *mixed* Riemann-Liouville fractional integral, which differs from the original in that the lower limit of the integral is a where $0 < a < x$. Fractional differentiation is considered as well as fractional integration in many dimensions. The final subsection is named "Other Special Cases of the **H**-Transformation", but the only case of marginal interest is one which leads to a transform with a kernel which is a mixture of those for the multidimensional Stieltjes and Hankel transforms.

The next section of Chapter 4 (§4.4) deals with Watson transforms, the name given to self-reciprocal multidimensional transforms of Mellin convolution type, i.e., transforms of the type (3) with the additional property that if $g = \mathbf{K}f$, then $f = \mathbf{K}g$. Kernels which satisfy this property also satisfy $k^*(\mathbf{s})k^*(\mathbf{1}-\mathbf{s}) = \mathbf{1}$ where k^* denotes the multidimensional Mellin transform of k. In general, however, if these conditions are satisfied, the transform of $k^*(\mathbf{s})$ does not exist. Nevertheless if we define a function $k_1(\mathbf{t})$ by the equation

$$k_1(t) = (2\pi i)^{-n} t^1 \underset{\mathbf{R}\to\boldsymbol{\infty}}{\text{l. i. m.}} \int_{\mathbf{1/2}-i\mathbf{R}}^{\mathbf{1/2}+i\mathbf{R}} \frac{k^*(\mathbf{s})}{(\mathbf{1}-\mathbf{s})^{\mathbf{1}}} \mathbf{t}^{-\mathbf{s}}\, d\mathbf{s},$$

then $k_1(\mathbf{t})$ exists and$\mathbf{t}^{-1}k_1(\mathbf{t}) \in L_2(\mathbb{R}^n_+)$, and we call $k_1(\mathbf{t})$ a *Watson kernel*; in terms of it we define the multidimensional *Watson transform* by

$$\mathbf{W}[f(\mathbf{t});\mathbf{x}] = \mathbf{D}\int_{\mathbb{R}^n_+} k_1(\mathbf{x}\cdot\mathbf{t})f(\mathbf{t})\mathbf{t}^{-\mathbf{1}}\, d\mathbf{t},$$

where $\mathbf{D}$ denotes the multidimensional differential operator $(\partial_1, \partial_2, \ldots, \partial_n)$ with $\partial_j = \partial x_j$. It is then shown that this operator $\mathbf{W}$ is self-reciprocal and unitary (although this latter concept has not been defined). Subsection 4.4.2 gives alternative conditions for the existence of a Watson kernel; and in §4.4.3 a largely featureless multidimensional Hankel transform is introduced, of which multidimensional sine and cosine transform are special cases. The next subsection (§4.4.4) entitled "Nonsymmetrical Formulas", considers the relation between two different Watson transforms when there is a simple relation connecting their kernels; this is interesting but not obviously applicable.

Section 4.5 is concerned with n-dimensional Bessel and Riesz potentials, their basic properties, and the function (and distribution) spaces to which they belong. The Riesz potentials are particularly important in applications in potential theory and elastostatics, so the account of their properties given here is most valuable; the only criticism I would have is that no reference is made to Riesz's classic paper [7]. Subsection 4.5.5 deals with hypersingular integrals where the order of the singularity in a Riesz potential is greater than the definition of the space over which it is defined, i.e., they are potentials of *negative* order. The final subsection is a very brief account of Riesz-Bessel potentials, special cases of which are respectively the Riesz and Bessel potentials; it appears to have no other properties of any interest.

The final section of this chapter, "Other Transformations", gives accounts of pyramidal fractional integrals, a particular nonconvolutional transformation (whose kernel contains a Lauracella function), and the Radon transform. The latter transform is of great interest to present-day workers in applied mathematics and theoretical physics, but it is hard to believe that the account given here will be of much use to them. What is the average "student of technology" to make of a definition in which an integral is taken over an element of P^n, "the space of all hyperplanes of $\mathbb{R}^n$ equipped with the usual topology" and whose integrand involves "the Euclidean measure on the hyperplane"? It would have been of more value to illustrate the definition by deriving a formula for the Radon transform in the special case $n = 3$ than by looking at the connection between the multidimensional Radon and Fourier transforms. The only useful aspect of this subsection is that it refers the reader to Helgason's monograph!

Again there is no subsection on "applications" in this chapter, almost certainly because the only multidimensional transforms appearing in the "applied" literature have been the Riesz potential and the Radon transform. It would have added greatly to the value of this book if there had been a section on the application of the Radon transform; that this does not appear is due, of course, to the fact that §4.6.3 is a totally inadequate basis for a discussion of even the simplest application.

The book is completed by two appendices, the first of which is a 75-page list of inverse two-dimensional Laplace transforms which does not appear to be superior to the corresponding table in (8), although the authors claim that most of those given here are new. The second appendix is a 28-page list of associated Laplace transforms.

In their preface to the book the authors state that it is "a cross between a textbook for students of mathematics, physics, and engineering and a monograph on this subject" but go on to say that "since this book is not a monograph, the references given at the end of this book are not complete." The list of references is fairly complete, though many of the books listed in it are on one-dimensional integral transforms, which I suppose is quite useful, since almost all of the multidimensional transforms treated in this book are "products" of identical one-dimensional transforms. There is even included in the references a paper on dual integral equations, although that particular subject is not featured in the book itself. The book seems to have been written in English by the authors themselves, since in the preface their only reference to this is to thank Mrs. M. Fritsch for "*checking* the English". The book is written neither in English—nor its American variant—though the statement of theorems, however ungrammatical, is unlikely to puzzle a mathematical reader. Considering the ethnic origins of the authors, the resulting language might have been picturesque, but it turned out to be bizarre. For this I blame not the authors but the publishers; if they were publishers of novels, their in-house editor would have put the language right.

Perhaps the authors were aiming too high in their assessment of their audience and restricted their field by omitting all references to so-called "mixed" transforms, which are transforms which are the "product" of a transform of one type for one subset of variables with a transform of another type for another subset of variables, for the applications of such transform are myriad. Their use has led to developments in pure mathematics—in dual integral equations, and through them in new operators of fractional integration and in the solution of mixed boundary value problems in theory of linear partial differential equations. That field is not yet exhausted; recently in a technological problem Hanson and Keer [9] used a Fourier transform coupled with a Kontorovitch-Lebedev transform. Their problem and most like it are solved by one-dimensional methods; this is true also of the problems discussed in the first two chapters of the present book, showing that the need has not yet arisen for a separate book on multidimensional integral transforms.

This book has several interesting features, but it is neither a suitable textbook for graduate students nor a comprehensive book of reference for research workers who use integral transforms in their investigations.

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Ian N. Sneddon
The University of Glasgow