

## BOOK REVIEW

*Hypo-analytic structures, local theory*, by Francois Treves, Princeton University Press, Princeton, NJ, 1992, xvii + 497 pp., \$65.00. ISBN 0-691-08744-X

Since the 1950s there has been a significant interplay between multivariable complex analysis and certain parts of partial differential equations theory. As is typical, problems of complex analysis have been treated by PDE methods, and in turn complex analysis has contributed concepts and driven certain lines of development in PDE theory. The literature is now enormous. Among the textbooks dealing specifically with this interplay we mention those by Morrey, Folland-Kohn, Ehrenpreis, several by Hormander, and some of the previous ones by Treves himself. His present book, which I shall refer to as “the text”, focuses on that part of the theory which may be characterized as arising from the study of real submanifolds of a complex manifold. Its emphasis is more on the basic local complex analytic and geometric foundations and concepts, which are rather diverse. It deals less with more advanced real analysis or functional analysis techniques. Thus, the “hypoanalytic” of the title is essentially a geometric concept, not to be confused with the more familiar term “hypoelliptic”, which is analytic in nature.

If we except the complex Monge-Ampere equation with its considerable success in the compact Kahler case, the predominant role of PDE theory in complex manifold theory is in the linear realm, via the Cauchy-Riemann equations. In the compact case the theory is wholly elliptic and well understood. In the noncompact case the theory is much more difficult and less complete. One must usually either deal with a boundary or place growth conditions at infinity. When the complex dimension exceeds one, the boundary of a bounded domain in complex space becomes of paramount importance. In favorable cases it is either a smooth real hypersurface or contains smooth real submanifolds. The trace of an analytic function on such a submanifold  $M$  is annihilated by all complex vectors of type  $(0, 1)$  tangent to  $M$ , i.e., is a CR function. The study of such functions is very important to the function theory of the domain.

Real submanifolds  $M$  may be divided into two classes: (i) those for which the tangent  $(0, 1)$  vectors form a vector subbundle (of constant rank) of the complexified tangent bundle of  $M$ , the CR submanifolds; and (ii) the rest, those with CR singularities. The text very wisely leaves aside case (ii), for which very little analytic theory has been developed, and begins with formally integrable, or involutive, structures, i.e., subbundles  $V$  of the complex tangent bundle of a real manifold which satisfy the Frobenius integrability condition (smooth sections of  $V$  are closed under Lie bracket). Associated to  $V$  is the natural deRham-Dolbeault complex with operator  $d''$ .

There are two main problems in this generality: (a) the integrability problem: find sufficiently many solutions with good properties to the homogeneous equations  $d''f = 0$ ; and (b) the inhomogeneous problem: solve  $d''f = g$  for the form  $f$ , given that the form  $g$  satisfies the compatibility condition  $d''g = 0$ , again with some control of  $f$ . Generally, the first problem is the more immediate and geometric one. The second is more analytic in nature and is used in solving the first. The ideal way of handling it is perhaps by means of a “homotopy formula”,  $g = d''Pg + Qd''g$ , where  $P$  and  $Q$  are manageable operators. For (domains on) complex manifolds the emphasis is usually global, e.g., as in solving the Levi problem. For more general involutive structures the primary emphasis in (a) is on the Complex Frobenius problem of existence of a complete, or maximally independent, set of local integrals. However, a general solution may not be a power series in these local integrals. This necessitates the finer concept of hypoanalytic structure: a covering of the manifold by open sets with integrals which are analytically related on the intersections.

The local integrability problem has ancient roots. The most important case concerns the existence of complex coordinates for an almost complex manifold. The one- (complex)dimensional case is the problem of isothermal coordinates or the local conformal mapping of a curved surface to the plane, a fundamental problem of cartography. It was solved in the real analytic case by Gauss in about 1820. The smooth case had to wait another one hundred years for the necessary analytic machinery to be developed. It was solved by Korn and Lichtenstein. By the late 1940s and early 1950s the concepts of complex and almost-complex manifolds were well established, and the higher-dimensional real analytic case was solved via the complex analytic Frobenius theorem. In 1957 Newlander and Nirenberg settled the smooth case. Their theorem, which is of fundamental importance in several regards, has since received several other proofs. One is given in the text based on the homotopy formula of Leray and Koppleman. The result is then extended to “elliptic” structures, i.e., those for which the annihilator of  $V$  contains no real one-forms. By a further extension the “rigid structures” studied by Rothschild and others is handled.

For more general structures the problem is much more subtle in the smooth case, with the formal integrability condition not always being sufficient. For the case of CR structures of nondegenerate real hypersurface type, counterexamples, due to Nirenberg and Jacobowitz-Treves, are given in the text. A major achievement here was the local CR embedding theorem of Kuranishi, extended by Akahori, for the strictly pseudoconvex case in seven or more dimensions. The 5-dimensional case still remains open. Currently, progress is being made on the regularity of the solution in yet-to-be-published work of Ma and Michel.

All these results rely on the solution of the inhomogeneous problem. In one complex variable it was first considered around the turn of the century, but it rarely occurs in the standard presentations of the subject. It has been ubiquitous in several variables since the work of Oka and Cartan. A landmark event was H. Lewy’s discovery in 1956 of his locally unsolvable equation. His operator is the tangential Cauchy-Riemann operator on the ordinary 3-sphere in the space of two complex variables. Another example was found by Mizohata and Grushin. It involves the operator in the  $xy$ -plane associated to the “folded complex structure”, where the basic analytic function is  $x + iy^2$ . Both operators play an important role in linear PDE theory. The text devotes a chapter to such examples of nonsolvability and nonintegrability, including the interesting phenomenon of one-sided integra-

bility discovered by C. D. Hill. Another chapter concerns necessary and sufficient conditions for a Poincaré lemma.

Chapter 2 presents one of the most important results of the theory, namely, the Baouendi-Treves approximation theorem, a very natural analogue of the Weierstrass approximation theorem. When there exists a complete set of first integrals, then every other solution to the homogeneous problem can be approximated uniformly on suitably small domains by polynomials in these integrals. It is both elementary to prove, following from a Fourier integral representation, and extremely useful. There are now many results which depend on it and many others which are greatly simplified by it. Local CR extension theorems are a good class of examples. It also underscores the appropriateness of the concept of hypoanalyticity.

Another strength of the text is a chapter on the FBI (Fourier-Bros-Iagolnitzer) transform. This is a generalization of the Fourier transform which is particularly suited to the study of hypoanalytic structures. The behavior of this transform bears on regularity properties, holomorphic extendability, and hypoanalyticity of distributions. It also is used to establish propagation of hypoanalyticity in certain cases. We are also assured that it plays a significant role in the microlocal theory. A final chapter deals with nonlinear equations. The main emphasis is on uniqueness in the Cauchy problem and on the approximation of not-so-smooth solutions by smoother ones.

My initial reaction, especially after learning from the preface that a second volume will be necessary for the microlocal theory, was that the text at 497 pages was perhaps too long. (I am always glad to find a book like this which contains information which I need to know, but I always hope to have to read it as little as possible!) After some experience with it, however, my opinion has changed. As the author warns us, the considerable amount of foundational material covered does not always make for the best sort of reading. However, many examples are spread throughout the book, and it is not clear that a shorter book would have been a better one. There are a few misprints (unavoidable in a work of this length), but none at all serious. The exposition is exceptionally clear and careful.

How far will the theory go and in what direction? It is risky to speculate, but it may be useful to consider some points. The theorem on isothermal coordinates and the Newlander-Nirenberg theorem have genuinely useful applications. But, while the real Frobenius theorem has a host of applications in elementary differential geometry, its complex analogue has scarcely any. It is more a theorem for its own sake at this stage. The technical difficulties of even the simplest proofs of the centrally important Kuranishi embedding theorem may indicate limits in this direction for the theory. Also, they show that even the simplest useful domains on CR manifolds tend to have characteristic points. Thus, the troublesome submanifolds with CR singularities must eventually be dealt with. In any event, it seems clear that this text of Treves will definitely play a positive role in future developments of the subject.

SIDNEY M. WEBSTER  
UNIVERSITY OF CHICAGO