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Lectures on nonlinear evolution equations, by Reinhard Racke. Friedrich Vieweg & Sohn Verlagsgesellschaft, Braunschweig, 1992, vii + 266 pp., DM 68. ISBN 3-528-06421-8

In the last decade the existence theory for global-in-time solutions of initial-value problems to fully nonlinear partial differential equations and systems has been strongly developed. Important contributions came from S. Klainerman and F. John. The clearly written book of R. Racke gives an introduction to this theory, presents many of the original results and proofs, and adds material that was not published before. The book is written on the level of advanced graduate students, but experts can also find interesting results.

The relevant ideas can best be explained by the example of the initial-value problem for the nonlinear wave equation

$$y_{tt}(t, x) - \Delta y(t, x) = \sum_{i,j=1}^{n} a_{ij}(y_t, \nabla y)\partial_i\partial_j y,$$
$$y(0, x) = y_0(x), \qquad y_t(0, x) = y_1(x),$$

where $x \in \mathbb{R}^n$, $t \geq 0$, and $a_{ij}(w) = 0(|w|^\alpha)$ for $|w| \to 0$ with $\alpha \geq 1$. It is well known that if the functions a_{ij} and the initial data y_0, y_1 are sufficiently regular, then a local classical solution of this problem always exists; but in general at a finite positive time t the solution u or the first or second derivatives of the solution become infinite, and consequently the classical solution cannot be continued beyond this blow-up time. For example, already in 1964 Lax [2] proved for the quasilinear wave equation in one space dimension that the second derivatives of the solution become infinite after a finite time (a shock wave is developing) and that this blow-up time is inversely proportional to the maximum-norm of the derivatives of the initial data.

The proof of the global existence results starts from this last observation, namely, that the smaller the classical solution is, the longer it exists; and it uses a second well-known result: The smaller the solution is, the closer it is to the solution of the linear wave equation. But the solution of the linear wave equation and its derivatives decay with the rate $t^{-(n-1)/2}$. One can hope that it is possible to combine both observations and to show that blow-up does not occur if the decay rate is large enough, which can be achieved by choosing the space dimension n large enough.

This program can be carried through using a priori estimates. The idea is to show that for the local solution a sufficiently high Sobolev norm stays bounded uniformly in time t, since the local solution can be continued as long as this Sobolev norm is finite. As a starting point let $D = (\partial_t, \nabla)$ and use the following well-known representation for the local solution $u(t) = Dy(t)$:

$$u(t) = Dw(t)y_1 + D\partial_t w(t)y_0 + \int_0^t Dw(t-r)f(r)\,dr, \tag{1}$$

with $f(t) = \sum_{i,j=1}^{n} a_{ij}(u)\partial_i\partial_j y$ and $(w(t)g)(x) := z(t, x)$, where z is the solution of

$$z_{tt} - \Delta z = 0,$$
$$z(0, x) = 0, \qquad z_t(0, x) = g(x).$$

A central observation is that $w(t)g$ satisfies the estimates

$$\|Dw(t)g\|_2 = \|g\|_2,$$
$$\|Dw(t)g\|_\infty \le c(1+t)^{-(n-1)/2}\|g\|_{n,1},$$

from which one obtains by interpolation and differentiation

$$\|Dw(t)g\|_{s,q} \le c(1+t)^{-(1-2/q)(n-1)/2}\|g\|_{s+N_p,p}$$

where $2 \le q \le \infty$, $1/p + 1/q = 1$, $N_p > n(1-2/q)$, and s is any nonnegative integer. Using this inequality to estimate the terms on the right-hand side of (1) yields

$$\text{(2)} \qquad \|u(t)\|_{s,q} \le c(1+t)^{-(1-2/q)(n-1)/2}(\|y_1\|_{s+N_p,p} + \|\nabla y_0\|_{s+N_p,p}$$
$$+ c\int_0^t (1+t-r)^{-(1-2/q)(n-1)/2}\|f(r)\|_{s+N_p,p}\,dr.$$

From this estimate one cannot conclude that $\|u(t)\|_{s,q}$ is uniformly bounded, since $f(r)$ depends on u. Therefore, to close the circle, two more estimates are needed: First, from the definition of f it follows by standard arguments for $s > [(s - N_p)/2]$ and $s_0 \ge s + N_p + 1$ that

$$\text{(3)} \qquad \|f(r)\|_{s+N_p,p} \le c\|u(r)\|_{s,2q/(q-2)}\|u(r)\|_{s_0,2}.$$

Second, the local solution satisfies the well-known energy estimate

$$\text{(4)} \qquad \|u(r)\|_{s_0,2} \le c\|u(0)\|_{s_0,2}\exp\left\{c\int_0^r \|Du(\tau)\|_\infty\,d\tau\right\}.$$

Now choose $q = 4$ and $s > \frac{n}{4} + 1$. Then $2q/(q-2) = q = 4$, and Sobolev's imbedding theorem yields $\|Du(\tau)\|_\infty \le c\|u(\tau)\|_{s,4}$. From (3) and (4) we thus obtain

$$\|f(r)\|_{s+N_p,p} \le c\|u(r)\|_{s,4}\|u(0)\|_{s_0,2}\exp\left\{c\int_0^r \|u(\tau)\|_{s,4}\,d\tau\right\}.$$

Insertion of this estimate in (2) yields an estimate, which can be used to show that $\|u(t)\|_{s,4}$ is bounded uniformly with respect to t, if the exponent $(1-2/q)(n-1)/2 = (n-1)/4$ in (2) is larger than 1, hence if $n > 5$. When $\alpha \ge 2$, then the inequality (3) is changed; and the same method yields existence of a global-in-time solution if $n \ge 3$.

This general idea can be used to prove existence of global-in-time solutions not only for the wave equation but for many nonlinear evolution equations; and besides studying the wave equation in detail, the author considers in this book the system of equations from elasticity, the heat equation, the equations of thermoelasticity, and the Schrödinger, Klein-Gordon, Maxwell, and plate equations. Of course, the results obtained differ for different evolution equations and depend on special properties of the equations. For example, for the wave equation Klainerman [1] developed a more powerful method which yields global existence for $n \geq 4$. This method is also presented in the book. Another interesting problem discussed in the book is presented by the equations of elasticity for cubic media, where flat points of the Fresnel surface cause a rate of decay smaller than for the equations of isotropic elasticity, where such flat points do not exist. This has consequences for the global existence results for the nonlinear equations, which are obvious from the ideas sketched above. It is known that the same problem appears for other first-order systems, for example, for Maxwell equations. Special attention in the book is given to the equations of nonlinear thermoelasticity, which is a coupled hyperbolic-diffusive system. For this system also a blow-up result is given which is proved by reducing the problem to a problem for genuinely nonlinear hyperbolic systems in one space dimension.

In summary, the book can be warmly recommended to anyone who wants a very readable introduction to this modern field.

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The admissible dual of GL(N) *via compact open subgroups*, by Colin J. Bushnell and Philip C. Kutzko. Princeton University Press, Princeton, NJ, 1993, ix + 313 pp., $59.50. ISBN 0-691-03256-4

To begin, consider the metric on the field of rational numbers $\mathbb{Q}$ defined as follows. Let p be a fixed prime number; if r is a fraction expressed in lowest terms, write it as $p^{n(r)}a/b$ where neither a nor b is divisible by p. If x, y are elements of $\mathbb{Q}$, define $|x-y|_p = p^{-n(x-y)}$; the completion of $\mathbb{Q}$ with respect to this metric yields a locally compact totally disconnected field which we denote by $\mathbb{Q}_p$. It contains a unique maximal compact subring $\mathbb{Z}_p$ and prime ideal $p\mathbb{Z}_p$, and $\mathbb{Z}_p/p\mathbb{Z}_p$ is naturally isomorphic with the finite field of p elements $\mathbb{F}_p$. $\mathbb{Q}_p$ is an example of a local nonarchimedean field; such fields originally arose via congruence relations in number theory, and their use