

BOOK REVIEW

The admissible dual of $GL(N)$ via compact open subgroups, by Colin J. Bushnell and Philip C. Kutzko. Princeton University Press, Princeton, NJ, 1993, ix + 313 pp., \$59.50. ISBN 0-691-03256-4

To begin, consider the metric on the field of rational numbers \mathbb{Q} defined as follows. Let p be a fixed prime number; if r is a fraction expressed in lowest terms, write it as $p^{n(r)}a/b$ where neither a nor b is divisible by p . If x, y are elements of \mathbb{Q} , define $|x - y|_p = p^{-n(x-y)}$; the completion of \mathbb{Q} with respect to this metric yields a locally compact totally disconnected field which we denote by \mathbb{Q}_p . It contains a unique maximal compact subring \mathbb{Z}_p and prime ideal $p\mathbb{Z}_p$, and $\mathbb{Z}_p/p\mathbb{Z}_p$ is naturally isomorphic with the finite field of p elements \mathbb{F}_p . \mathbb{Q}_p is an example of a local nonarchimedean field; such fields originally arose via congruence relations in number theory, and their use is now commonplace in algebraic number theory. In everything that we say below, F will denote such a field, \mathfrak{o} its maximal compact subring with prime ideal \mathfrak{p} , and $\kappa = \mathfrak{o}/\mathfrak{p}$ its (finite) residue field of characteristic p .

Now let $G = \mathbb{G}L_n$ denote the group of invertible $n \times n$ matrices with entries in F . It inherits a topology from $F^{(n^2)}$ which makes it into a locally compact totally disconnected topological group which is second countable (in brief, a *t.d. group*). In particular, G has lots of compact open subgroups. By a *representation* of G we shall mean a pair (π, V) where V is a complex vector space and π is a homomorphism: $G \rightarrow \text{Aut}(V)$ of abstract groups. (We note that V need not be finite dimensional.) The representation (π, V) is *irreducible* if it has no proper nontrivial invariant subspaces. We say that (π, V) is *smooth* if for any $v \in V$, $\text{Stab}_G(v) = \{g \in G \mid \pi(g)v = v\}$ is an open subgroup; we say that (π, V) is *admissible* if it is smooth and if, for any compact open subgroup K , the space of invariants V^K is finite dimensional. (This latter notion has its origins in the theory of automorphic forms.) We denote by G^\wedge the collection of isomorphism classes of irreducible admissible representations of G ; in this review we shall use the word *representation* to mean an element of G^\wedge unless otherwise stated.

This brings us to the subject matter of the book by Bushnell and Kutzko.

Problem. Classify G^\wedge .

The book under review provides a complete algebraic classification of G^\wedge , and it is a milestone in this subject. Before describing it, we provide some historical background which reflects the reviewer's own prejudices.

As an example, and to help the reader see that this might be a worthwhile problem, we note that if $n = 1$ one can show that G^\wedge consists of continuous homomorphisms from F^\times to \mathbb{C}^\times and is intimately bound up with classical reciprocity laws in abelian class field theory.

Mautner [Ma] was the first person to undertake the study of (unitary) representations of $\mathbb{G}\mathbb{L}_2$; he appears to have been motivated from the point of view that such groups provide examples of abstract harmonic analysis quite different from that of real Lie groups. Here is such an example.

Let $K = \mathbb{G}\mathbb{L}_n(\mathfrak{o})$; this is a compact open subgroup of G , and it contains a normal pro- p subgroup $U = \{x \in K \mid x - 1 \text{ has entries in } \mathfrak{p}\}$. One verifies that $K/U \simeq \mathbb{G}\mathbb{L}_n(\kappa)$. Let (σ, W) be an irreducible cuspidal representation of $\mathbb{G}\mathbb{L}_n(\kappa)$. (Such representations were classified by J. A. Green.) This inflates to a representation of K ; take any irreducible extension $\tilde{\sigma}$ of it on $K^+ = F^\times K$. (Here we are identifying F with scalar matrices.) Now let

$$V = \{f: G \rightarrow W \mid f \text{ locally constant, compact support,} \\ f(kx) = \tilde{\sigma}(k)f(x), k \in K^+\}.$$

G acts on V by right translations. The representation (π, V) has the following nonobvious properties:

- (i) It is irreducible and admissible.
- (ii) It has matrix coefficients which are compactly supported modulo the centre of G .

Property (ii) implies that (π, V) is *essentially square integrable*, i.e., it has matrix coefficients which are square integrable after a twist by a one-dimensional representation of G . In the traditional theory of noncompact real reductive Lie groups the classification of representations of this sort is a cornerstone of the theory, but they *never* have compactly supported matrix coefficients. (We remark that G also has essentially square integrable representations which do not have property (ii).) The example above was produced by Mautner for particular $\tilde{\sigma}$. For the purposes of this review we shall call any representation of G satisfying properties (i) and (ii) *supercuspidal*. A representation which has square integrable matrix coefficients is said to belong to the *discrete series*.

The subject received considerable impetus when R. P. Langlands laid down a broad hypothetical framework for nonabelian class field theory (both local and global) in the latter half of the 1960s. In this framework there should be a bijection (satisfying certain arithmetic compatibility conditions) between G^\wedge and the set of n -dimensional complex representations of a variant of the Galois group of F . In particular, irreducible supercuspidal representations should correspond to irreducible n -dimensional representations of this group.

Given any reasonable category of representations for (the F -valued points of) a reductive group over some field F , there is a standard way to build lots of representations starting from F -rational Levi components of parabolic subgroups. In the present setup this proceeds as follows. One starts with a partition $n_1 + n_2 + \cdots + n_r = n$. The group $M = \mathbb{G}\mathbb{L}_{n_1} \times \cdots \times \mathbb{G}\mathbb{L}_{n_r}$ embeds down the diagonal of $\mathbb{G}\mathbb{L}_n$ in the obvious way; let P be the group whose entries are zero below these blocks. Let π_i be an irreducible supercuspidal representation of $\mathbb{G}\mathbb{L}_{n_i}$; we form the tensor product of the π_i and inflate it to P via the obvious map $P \rightarrow M$. If one induces (in

a suitable sense) this representation to G , the result is a smooth representation of G which has a finite composition series whose factors are admissible. P is an example of a parabolic subgroup in G ; the process above is *parabolic induction*. Parabolic induction has been studied intensively by many people in a more general framework; it does not exhaust the admissible dual. In fact, Jacquet proved that the missing representations are precisely the supercuspidal representations of G ; in the late 1970s Bernstein and Zelevinski gave a precise description of G^\wedge modulo a complete knowledge of supercuspidal representations. For an account of this we refer the reader to [R].

One early method to directly construct supercuspidal representations of G involved taking some class of “regular” characters on compact maximal tori (= separable field extensions of degree n , suitably embedded in G) and constructing suitable representations of G via induction from open compact mod centre subgroups. This approach was motivated by the construction of discrete series representations for real Lie groups, but in retrospect, it was doomed ultimately to fail, although it had some striking successes (notably if $(p, n) = 1$; see [H1]). A philosophical explanation is that the arithmetic of real Lie groups is too simple to use as a model for the arithmetic of reductive p -adic groups. A serious shortcoming with such methods was that they never provided any intrinsic way of knowing how many supercuspidals one had constructed.

Bushnell and Kutzko follow an alternative philosophy, which is to study G^\wedge by restriction to compact open subgroups and to identify supercuspidal representations as those which contain a particular class of representations of suitable compact open subgroups. This idea can be traced back to Mautner’s original work but was emphasized explicitly and clearly by Roger Howe. In [H2] he defined the notion of “essential K -type”, where K is a subgroup of the form $K_m = I + \mathbb{M}_n(\mathfrak{p}^m)$, $m \geq 0$. He showed that any element of G^\wedge contained an essential K -type for some K , and from this he was able to recover independently a result of Casselman that any finitely generated admissible representation of G has a finite composition series. (Casselman’s result is valid more generally.) Using essential K -types, he was able to describe roughly a large part of G^\wedge , including many supercuspidal representations, and to show that the character of an irreducible admissible representation of G was a locally constant function on the set of regular elements of G . Slightly later Kutzko [K] constructed supercuspidals of GL_2 by studying how elements of G^\wedge behave on restriction to compact open subgroups; using this method, he could show that he had produced all of them. Further progress was made by Carayol in his thesis [Ca]. He considered irreducible admissible representations of G which contain *s-cuspidal* representations. These are certain finite-dimensional representations of congruence subgroups of parahoric subgroups (see below) of G ; Carayol proved that any element of G^\wedge which contains such a representation is supercuspidal and can be explicitly exhibited as an induced representation from a compact mod centre open subgroup of G . Using global methods (trace formula), Carayol was further able to show that if n is prime, then *every* supercuspidal representation of G arises in this manner.

The “restriction” approach can be understood in the following framework. Let G be any t.d. group, K a compact open subgroup of G , and (σ, W) an irreducible admissible representation of K . (σ is a fortiori finite dimensional.) Then the elements of G^\wedge which contain σ on restriction to K are in bijection with the isomorphism classes of simple modules of a certain associative algebra $\mathcal{H}(K, \sigma)$ which we define as follows. Let (σ^\vee, W^*) denote the contragredient representation of σ on K (i.e.,

W^* is the dual space of W and $(\sigma^\vee(k)f)(w) = f(\sigma(k^{-1})w)$ if $f \in W^*$, $w \in W$. Then

$$\mathcal{H}(K, \sigma) = \{f: G \rightarrow \text{End}(W^*) \mid f \text{ compactly supported,} \\ f(kgk') = \sigma^\vee(k)f(g)\sigma^\vee(k'), k, k' \in K\}.$$

Addition is defined in the obvious way, and multiplication is defined by convolution; moreover, $\mathcal{H}(K, \sigma)$ has an identity element e :

$$e(x) = \begin{cases} (\dim W)^{-1} \sigma^\vee(x) & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Note that an element of $\mathcal{H}(K, \sigma)$ is determined by its values on a finite collection of double cosets. The collection of all double cosets which support nontrivial elements of $\mathcal{H}(K, \sigma)$ is called the *intertwining* of σ . We shall call such an algebra an *intertwining algebra*.

One early application of this theme was the paper [H1]. There Howe was able to show exhaustion of supercuspidals for $\mathbb{G}\mathbb{L}_2$ by proving that certain intertwining algebras related to Whittaker models are abelian and then by employing the Plancherel formula; this is an idea which merits dusting off.

Howe focused on this theme again in a 1983 summer workshop at Chicago. (See [HM1].) There he indicated by examples that for some K, σ (when $G = \mathbb{G}\mathbb{L}_n$) one could interpret $\mathcal{H}(K, \sigma)$ as an intertwining algebra for a product of *smaller* $\mathbb{G}\mathbb{L}_n$'s defined over finite extensions of F . Independently (and perhaps more significantly for Bushnell and Kutzko), Waldspurger [W] elegantly combined the techniques of [H1] and [Ca], pushed them to their absolute limit, and simultaneously proved that many intertwining algebras could be interpreted in terms of another well-studied algebra \mathcal{H} . This algebra was introduced by Iwahori and Matsumoto [IM] in the context of reductive p -adic groups; we define it because it plays a central role in the book under review.

If H is the group of F -valued points of any reductive group defined over F , then H is naturally a t.d. group. (Our G above is a special case.) Further, H always has a distinguished class of compact open subgroups (*parahoric* subgroups). In particular, it contains a distinguished conjugacy class of minimal parahoric subgroups called *Iwahori* subgroups. For G above one can take an Iwahori to be the subgroup whose entries on or above the diagonal are in \mathfrak{o} and whose entries below the diagonal are in \mathfrak{p} ; it is a Borel group “mod \mathfrak{p} ” in the group denoted K earlier. Let \mathcal{H} be the algebra $\mathcal{H}(K, 1)$ where K is an Iwahori subgroup; this algebra can be described explicitly by generators and relations (in terms of the affine Weyl group of H); its module category is relatively well understood for a wide class of H (in particular, G) by virtue of fundamental and penetrating work of Borel [B] and Kazhdan and Lusztig [KL]. We call it an *affine Hecke algebra*.

The fundamental results in the book by Bushnell and Kutzko are in the spirit of Waldspurger, but this book goes far beyond preexisting techniques, in large part by introducing completely new ideas. In particular, it produces the (many) missing supercuspidals and shows that all supercuspidals have been obtained. The starting point is the theorem (conjectured by Allen Moy [M] and proved by Bushnell [Bu], and also by Howe and Moy [HM2]) that any irreducible admissible representation of G contains a *fundamental G -stratum*. This is a much more general and flexible

version of essential K -type due to Allen Moy; to start, one replaces the groups K_m mentioned earlier by congruence subgroups of *arbitrary* parahoric subgroups. Briefly, the book can be divided into three parts:

- (i) the notion of a simple stratum and its properties (Chapters 1 and 2);
- (ii) from this, the development of “simple” characters (Chapter 3), culminating in the all-important definition of a *simple type* (Chapter 5) and resulting intertwining/Hecke algebra isomorphisms;
- (iii) applications to representation theory (Chapters 6, 7, 8).

The hardest work is done in Chapters 2 and 3. A “level 0” simple stratum, which is a purely arithmetic object attached to G , gives rise to two families (each ordered by inclusion) of compact open subgroups of G (3.1). This leads to the notion of sets of *simple characters* defined on the first family; a priori this depends on the choice of a defining sequence (2.4). The authors take care to show that such a set is nonempty and give an inductive method for describing its contents (3.2–3.3); in 3.3 they show that in fact the definition depends only on the equivalence class of simple stratum by studying the intertwining of these characters (3.3.2).

With the definitions and properties of simple characters well established, the authors then carefully show (5.1, 5.2) how such characters extend to finite-dimensional representations of the top member J of the second family of compact open subgroups. Thanks to their earlier results (1.5, 2.6, 3.3) on intertwining, the intertwining of these new representations is easily described in terms of J and a smaller reductive subgroup. In 5.5.10 the authors define the key notion of *simple type*: roughly speaking, it is an extension as above tensored with a cuspidal representation of a certain reductive group defined over a finite field which inflates to a representation of J . One requires that the finite reductive group be the e -fold product of GL_f for certain e, f .

In fact, the simple type *par excellence* is the trivial representation of an Iwahori subgroup. The main result (5.6.6) in the book under review is that the intertwining algebra of a simple type is isomorphic to an affine Hecke algebra; roughly speaking, the size of the algebra is controlled by the size of the finite reductive group.

The authors then apply this machinery to study G^\wedge . The results are impressive. In short order they show in Chapter 6 that any supercuspidal containing a simple type must be induced from a compact mod centre open subgroup (closely related to J above), and then that the simple type is severely restricted. Moreover, any other element of G^\wedge containing that type is also supercuspidal; if a supercuspidal contains two simple types, those types are conjugate in G . The authors complete their study of admissible representations containing a simple type in Chapter 7 by relating their methods with the Jacquet functor. This enables them to study such representations in a manner reminiscent of Borel [B] and (especially) of Waldspurger, from whose paper [W] they acknowledge inspiration. In particular, they obtain formulae for the formal degrees of discrete series containing a simple type (7.7), thus generalizing work of Waldspurger and others.

In Chapter 8 the authors define *split types*; the main result here is that any representation which does not contain a simple type must contain a split type, and then that such a representation has at least one nontrivial Jacquet module. From this it is immediate (via a result of Jacquet alluded to earlier) that a supercuspidal representation must contain a simple type, thus completing Chapter 6. The authors then explicitly classify representations containing a split type; in particular, they

show that any discrete series must contain a simple type, thus completing Chapter 7.

We now try to add further perspective to this bowdlerized précis. First, various workers have previously constructed supercuspidals or studied intertwining algebras by proceeding “one level at a time” or “halfway” up congruence subgroups. The authors completely avoid this by using the rather subtle notion of simple character sets and then defining a type on a previously well-defined subgroup. This requires great care in its construction and definition, but it has the advantage that it is the right construction. Indeed, with hindsight it seems unlikely that any earlier methods could have been pushed to work in complete generality. One also sees very clearly that in general there is *no hope* of parametrizing supercuspidal representations by characters of compact mod centre maximal tori.

Second, as a corollary of the authors’ approach, one obtains classification results in complete generality, whereas earlier results were always for special cases (n the product of 2 primes, or $(n, p) = 1, \dots$), or they were modulo the knowledge of supercuspidals; not infrequently people have relied on global techniques from the theory of automorphic forms to prove exhaustion. Here we have powerful new conceptual techniques which do not rely on global methods at all; moreover, supercuspidal representations arise naturally as those containing a special kind of simple type, and one *independently* recovers a more arithmetic form of the classification of Bernstein and Zelevinski.

The authors have made a considerable effort to provide precision and complete proofs, but this book is probably not easy for a beginner to read. Fortunately, they provide a road map at the end of their introduction (recommended reading) which tells the reader how to arrive at the applications in the most painless fashion possible. The reviewer also recommends [He] for an introduction to the contents of this book and to other recent developments in this subject.

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