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BOOK REVIEW

Introduction to multidimensional integrable equations, by B. G. Konopelchenko. Plenum Publishing, New York, 1992, x + 292 pp, $85.00. ISBN 0-306-44220-5

The field of soliton theory and the related methods of solution of the underlying nonlinear evolution equations, referred to as the Inverse Scattering Transform (IST), have been intensively studied for over twenty years. There are many nonlinear equations of physical significance which fit into the theory, the best-known examples being the Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS), and sine-Gordon equations. These equations as well as many others which are $(1 + 1)$-dimensional (one space-one time) are special in that their exact solution, corresponding to initial data on the infinite line which decay sufficiently rapidly, reduces to a study of linear equations. The essential methods were developed in the 1970s. In the seventies it was known that certain $2 + 1$ equations also had properties which suggested that the IST techniques would be applicable. The prototypical equations are the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations, which arise in physics and are $2 + 1$ extensions of the KdV and NLS equations. Precisely how to solve these and related multidimensional systems is a central topic of this book.

In Chapter 1 a brief review of the IST method and related considerations is outlined. The solution process for 1+1 problems is outlined, and indications of what will be covered in later chapters is given. For example, in pages 16–18 the solution of the KdV equation is described, and important results are summarized. Most likely readers will need to supplement this part of the text with any one of a number of books which describe the IST method for $1 + 1$ problems in considerable detail (this book has numerous references—over 700 of them—which will aid, though they might exhaust, the interested reader). The critical aspect of $1 + 1$ problems is that their solution is intimately connected to the study of local Riemann-Hilbert (RH) boundary value problems. In Chapter 2, probably the key chapter in the book, the methods applicable to $2 + 1$ problems are explained primarily in terms of the prototypes, KP and DS. The main new features of $2 + 1$ problems, corresponding to decaying data in the plane, are derived. Specifically, the reader finds out that, depending on the signs of the terms in KP or DS, analysis leads to the study of either nonlocal RH problems or DBAR problems. DBAR problems in complex analysis have been intensively studied by mathematicians. The application of these ideas in nonlinear wave problems is undoubtedly notable. The point of view here is one of explaining the methods and not going into the detailed functional analysis—often typical of applied mathematics. In recent years much of (but not all) the necessary functional analysis has been developed which has put the methods described in this book on a reasonably firm footing.

Chapters 3 and 4 contain a selection of techniques either to obtain special solutions, develop insight into the structure of the hierarchies of equations using algebraic analysis (sometimes referred to as the "tau function" approach), or understand the underlying symmetry structure of the equations. In Chapter 5 a discussion of the solution of higher-dimensional nonlinear equations is considered. The well-known prototype here is the four-dimensional self-dual Yang-Mills (SDYM) system, and methods to obtain solutions corresponding to specific gauge choice (SU(2)) are given. Actually recent work (not described in this book) has shown that the SDYM system, with the freedom of gauge heavily used, is actually a "master system" from which all of the well-known soliton equations can be obtained, by reduction, as special cases. On the other hand, the SDYM system seems to be in a very small class of integrable systems of high dimension (beyond the three-dimensional problems described earlier). The author explains in Chapter 5 why the scattering theory associated with multidimensional problems puts severe constraints on the evolution of data and why this is a serious obstacle to finding flows which will be compatible with linear systems for which the IST technique can be implemented.

The broad field of integrable systems is now enormous. No book could possibly cover all of the important aspects. This book carves out a piece of territory, multidimensional integrable equations and aspects of their solutions, especially those corresponding to decaying initial data, employing the IST method. The book can serve a reader well in many ways. The methods are explained, with lots of references given. Hence interested researchers can access the necessary external papers and books for any required material that is not contained in this book. The book would be a useful addition to a library's collection in a field of study which continues to expand rapidly.

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