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Theory of entire and meromorphic functions. Deficient and asymptotic values and singular directions, by Zhang Guan-Hou. Translations of Mathematical Monographs, vol. 122, American Mathematical Society, Providence, RI, 1993, xi + 375 pp., \$182.00. ISBN 0-8218-4589-6

INTRODUCTION

An entire function is a holomorphic mapping $\mathcal{C} \rightarrow \mathcal{C}$. A function is meromorphic in the domain D , if it is holomorphic in D , except for poles. If we speak of a meromorphic function without specifying D , it is understood that it is meromorphic in \mathcal{C} . A *value* is a point on the Riemann sphere $\hat{\mathcal{C}}$.

The *Fundamental theorem of algebra* (Gauss's doctoral thesis, 1799) is the oldest result on value distribution. An easy consequence of it is the fact that a polynomial of degree n has n complex roots (counting with proper multiplicity). The entire function $\exp(z)$ behaves in a very different fashion: It omits the values zero and infinity entirely and takes every other value infinitely often.

What can be said about the distribution of values of entire and meromorphic functions in general? This is the subject of Nevanlinna Theory. The name is chosen in honor of the brothers F. and R. Nevanlinna, who developed the theory in the early 1920s. H. Weyl called the appearance of R. Nevanlinna's first paper on the subject "one of the few great mathematical events of the century". The field continues to be active: a bibliography compiled by A. A. Gol'dberg covering the period 1953–1971 and the excellent survey article [GLO2] contain together about 1,000 citations. Nevanlinna Theory has important applications in other branches of mathematics ranging from the theory of transcendental numbers to probability theory and statistics and to theoretical physics.

This review, like the book under review, will restrict itself to the following items:

1. Basic definitions
2. Nevanlinna Theory before the Nevanlinnas
3. The characteristic function
4. The first Fundamental Theorem (1.FMT)
5. The second Fundamental Theorem (2.FMT)
6. Defects and the Defect Relation
7. Borel directions, Julia lines and lines of accumulation
8. Asymptotic values
9. Further work on Nevanlinna Theory

1. BASIC DEFINITIONS

We shall need some standard notation.

$n(r, a, f)$ = number of a -points of $f(z)$ in the disk $\{z : |z| \leq r\}$

(each point counted with its proper multiplicity).

The counting function of the a -points of f is

$$N(r, a, f) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r.$$

The proximity function of ∞ with respect to f is

$$m(r, \infty, f) = (1/2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ |x| = \sup(\log |x|, 0)$.

The proximity function of a with respect to f is

$$m(r, a, f) = m(r, \infty, 1/(f - a)).$$

The Maximum Modulus of f is

$$M(r, f) = \sup_{|z|=r} |f(z)|.$$

An entire function is of order λ , if, for every $\epsilon > 0$, $\log^+ M(r, f) = O(r^{\lambda+\epsilon})$ as $r \rightarrow \infty$.

2. NEVANLINNA THEORY BEFORE THE NEVANLINNAS

In 1879 E. Picard published his stunning

Picard's Theorem. *A meromorphic function takes every value, with at most two exceptions, an infinite number of times.*

Picard's proof used the elliptic modular function; it was by no means elementary.

Further progress was made possible by Weierstrass's canonical product representation of entire functions (1876). In 1892 J. Hadamard proved a representation formula for entire functions of finite order from which it follows that $N(r, 0, f)$ and $\log^+ M(r, f)$ are of the same order of magnitude, except possibly in the case of functions of integer order p with $\log M(r, f) \approx Ar^p$ for some positive A . His proof needed difficult estimates of canonical products.

E. Borel found a proof of Picard's Theorem for entire functions in 1899 using only fairly simple properties of canonical products. He extended it to meromorphic functions four years later. The crucial point was again the fact that for an entire function $N(r, a, f)$ and $\log M(r, f)$ are of the same order of magnitude for all complex a with at most one exception.

In 1899 J. L. Jensen published a paper with the accurate, even if not modest, title *A new and important theorem in function theory*. In the notation introduced above, the theorem can be stated as

$$(a) \quad \log |f(0)| = m(r, \infty, f) - m(r, 0, f) + N(r, \infty, f) - N(r, 0, f)$$

(slight modification, if $f(0) = 0$ or ∞).

3. THE CHARACTERISTIC FUNCTION

The Nevanlinnas rewrote (a) as

$$T(r, f) \stackrel{\text{def}}{=} N(r, \infty, f) + m(r, \infty, f) = N(r, 0, f) + m(r, 0, f) + O(1)$$

as $r \rightarrow \infty$.

$T(r, f)$ is the *characteristic function* of f .

The characteristic function is an increasing, convex function of $\log r$. If $T(r, f) = O(\log r)$ as $r \rightarrow \infty$, then f is a rational function.

The *order* λ and the *lower order* μ of the meromorphic function f are defined by

$$\left. \begin{aligned} \lambda &= \limsup \frac{\log T(r, f)}{\log r} \\ \mu &= \liminf \frac{\log T(r, f)}{\log r} \end{aligned} \right\} \quad (r \rightarrow \infty).$$

The maximum modulus of a meromorphic function can behave in a very irregular way, and the lack of a suitable comparison function had been a major obstacle in the theory of value distribution.

L. Ahlfors and, independently, T. Shimizu (1921) gave an important geometrical interpretation of the characteristic function. The inverse function $z = f^{-1}(w)$ has a Riemann surface \mathcal{S} . One can regard w as a point on the Riemann sphere $\hat{\mathbb{C}}_0$ and \mathcal{S} as a covering surface of $\hat{\mathbb{C}}_0$. Let $S(t)$ be the area of \mathcal{S} lying above the spherical cap $\{z : |z| < t\}$ of $\hat{\mathbb{C}}_0$. $S(t)$ is calculated in the Euclidean metric of $\hat{\mathbb{C}}_0$. Then

$$T_0(r, f) = \frac{1}{\pi} \int_0^r \frac{S(u)}{u} du = T(r, f) + O(1)$$

as $r \rightarrow \infty$. Terms that remain bounded as $r \rightarrow \infty$ are of no importance in Nevanlinna theory, so $T_0(r, f)$ can be used in place of $T(r, f)$. The convexity properties of T mentioned above are geometrically evident for T_0 .

By writing Jensen's Formula with $f - a$ in place of f , it is easy to prove the First Fundamental Theorem.

4. THE FIRST FUNDAMENTAL THEOREM

The First Fundamental Theorem (1.FMT). For all values a

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(1)$$

as $r \rightarrow \infty$.

We look at $f(z) = \exp(z)$.

$$\left. \begin{aligned} T(r, f) &= m(r, \infty, f) \\ m(r, \infty, f) &= r/\pi \\ N(r, a, f) &\approx r/\pi \end{aligned} \right\} \quad (a \in \mathbb{C}, a \neq 0).$$

In this example the counting function $N(r, a, f)$ is of the order of magnitude of $T(r, f)$ for all complex $a \neq 0$, while for $a = 0$ or ∞ the proximity function $m(r, a, f) = T(r, f)$. This illustrates the dichotomy expressed by the 1.FMT: Either $N(r, a, f)$ is large, in which case the function must have many zeros in the disk $\{z : |z| < r\}$, or $m(r, a, f)$ must be large. This is only possible if $f(z)$ is close to the value a on a substantial part of $\{z : |z| = r\}$. For all meromorphic functions $N(r, a, f)$ preponderates for a vast majority of a . This is shown by the Second Fundamental Theorem.

5. THE SECOND FUNDAMENTAL THEOREM

The Second Fundamental Theorem (2.FMT). Let $f(z)$ be a nonconstant meromorphic function, and let a_j ($j = 1, 2, \dots, p$, $p > 2$) be distinct values. Then

$$(b) \quad (p - 2 + o(1)) T(r, f) < \sum_1^p \bar{N}(r, a_j, f) \quad (r \rightarrow \infty) \quad ||.$$

The symbol $||$ indicates that r must tend to ∞ , skipping a set of finite Lebesgue measure, if f is of infinite order. Here $\bar{N}(r, a, f)$ is the counting function of the a -points disregarding their multiplicity.

Picard's Theorem is an immediate consequence of the 2.FMT, since it shows that \bar{N} cannot equal zero for three a . But the 2.FMT contains much more precise information.

6. DEFECTS AND THE DEFECT RELATION

Replacing the term $pT(r, f)$ on the left-hand side of (b) by the expression $\sum_1^p (m(r, a_j, f) + N(r, a_j, f))$, we can rewrite (b) as

$$(*) \sum_1^p \{m(r, a_j, f) + [N(r, a_j, f) - \bar{N}(r, a_j, f)]\} + o(T(r, f)) \leq 2T(r, f) \quad ||.$$

We define the *deficiency* of a with respect to f as

$$\delta(a, f) = \liminf_{r \rightarrow \infty} \frac{m(r, a, f)}{T(r, f)},$$

and the *ramification defect* of a with respect to f as

$$\epsilon(a, f) = \liminf_{r \rightarrow \infty} \frac{N(r, a, f) - \bar{N}(r, a, f)}{T(r, f)}.$$

By (*), for any finite set A of values,

$$(c) \quad \sum_{a \in A} [\delta(a, f) + \epsilon(a, f)] \leq 2.$$

We call a *deficient* with respect to f if $\delta(a, f) > 0$; we call a *ramified* with respect to f if $\epsilon(a, f) > 0$. We call a *defective* if it is either deficient or ramified.

We call

$$\Delta(a, f) = \limsup_{r \rightarrow \infty} \frac{m(r, a, f)}{T(r, f)}$$

the *Valiron defect* of a with respect to f .

It follows from (c) that the set of defective values is countable and that the *defect relation*

$$(d) \quad \sum_{a \in \hat{\mathcal{C}}} [\delta(a, f) + \epsilon(a, f)] \leq 2$$

holds.

$\delta(a, f) = 1$ for an omitted value and always $0 \leq \delta(a, f) \leq 1$, by the 1.FMT. Also $0 \leq \epsilon(a, f) \leq 1$. Picard's Theorem is therefore a simple consequence of the defect relation. But the notion of defect allows one to make

much finer distinctions between the distributions of different values than was possible prior to its introduction.

The defect relation is the only restriction on the deficiencies and ramification defects of meromorphic functions. D. Drasin proved in a very ingenious, long, and difficult paper (1977) the

Inverse Theorem of Nevanlinna Theory. *Given a countable set of values $\{a_j\}$, numbers δ_j and ϵ_j subject to the restrictions $0 \leq \delta_j \leq 1$, $0 \leq \epsilon_j \leq 1$, and $\sum[\delta_j + \epsilon_j] \leq 2$, there is a meromorphic function f whose defective values are the a_j with $\delta(a_j, f) = \delta_j$, $\epsilon(a_j, f) = \epsilon_j$.*

A simpler proof of the theorem is due to A. Eremenko and M. L. Sodin (1990).

The functions f used in the proof of the Inverse Theorem of Value Distribution are of infinite order. For functions of finite order, or just of finite lower order, there are many restrictions on the deficiencies. Their investigation is a big, and still not completely finished, chapter of Nevanlinna Theory. For example,

Theorem [W. K. Hayman (1964)]. *Let f be a meromorphic function of finite lower order. If $\alpha > 1/3$, then $\sum_{a \in \widehat{\mathbb{C}}} \delta(a, f)^\alpha < \infty$.*

The defect relation is an inequality. Many papers have studied the nature of the functions of finite order for which

$$(e) \quad \sum \delta(a, f) = 2.$$

Theorem [D. Drasin (1974)]. *If $f(z)$ is a meromorphic function of order $\lambda < \infty$ for which (d) holds, then*

- (1) λ is an integer multiple of $1/2$ and $\lambda \geq 1$;
- (2) there are at most 2λ deficient values;
- (3) each deficient value is an asymptotic value¹.

Actually, more can be said about the behavior of f . In a recent paper Eremenko proves the full truth under a weaker hypothesis than (e).

R. Nevanlinna's proof of the 2.MFT was based on an estimate of $m(r, f'/f)$. F. Nevanlinna gave a second proof by putting a distribution of mass ν on the Riemann sphere, lifting it to the Riemann surface of the inverse function $z = f^{-1}(w)$ considered as a covering surface of the Riemann sphere and integrating over the surface. The choice of ν is motivated by a study of the singularities of the function which maps the universal covering surface of the sphere with punctures at p points onto one of the unit disk, the sphere, or the plane. (By the uniformization theorem exactly one of these mappings exists.) F. Nevanlinna's investigations inspired L. Ahlfors to build his famous theory of covering surfaces. This theory leads to a theory of value distribution for a large class of functions containing the meromorphic functions. J. Miles showed (1969) that the 2.FMT in the form given above is actually a consequence of Ahlfors's theory.

To conclude this section, I quote two theorems which show the striking conclusions that can be drawn from the FMTs.

¹ The value a is asymptotic if there is a path Γ tending to infinity such that $f(z) \rightarrow a$ as z tends to infinity on Γ .

Theorem [R. Nevanlinna (1939)]. *Let $f(z)$ and $g(z)$ be two meromorphic functions. If for five values of a $[f(z) = a] \leftrightarrow [g(z) = a]$, without regard to multiplicity, then $f(z) \equiv g(z)$.*

The five values of a of the theorem cannot be reduced to four, as is shown by $f(z) = \exp(z)$, $g(z) = \exp(-z)$.

Theorem [J. Miles and J. Rossi (1992)]. *Let $f(z)$ be a transcendental entire function satisfying the differential equation*

$$y^{(k)} + A y^{(k-1)} + \sum_0^{k-2} B_j y^{(j)} = 0,$$

where the A and B are entire functions with orders $\mu(A) = 1/2$, $\mu(B) < 1/2$. Then

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, a, f)}{\log r} = \infty$$

for all $a \in \mathcal{C}$.

7. BOREL DIRECTIONS, JULIA LINES, AND LINES OF ACCUMULATION

The counting function $N(r, a, f)$ depends only on the radial distribution of the a -points. It is therefore surprising that Nevanlinna Theory is capable of giving interesting information about the distribution of the arguments of the a -points. The earliest results in this direction go back to the 1920s and are due to E. Borel, G. Julia, and H. Milloux. The following notation and definitions will be used:

$$\Delta(\beta, \eta) = \{z : \beta \leq \arg z \leq \eta\};$$

for simplicity we write $\Delta(\alpha)$ for $\Delta(\alpha, \alpha)$.

$n(r, \alpha, \epsilon; f)$ is the number of zeros of f in $\Delta(\alpha - \epsilon, \alpha + \epsilon) \cap \{z : |z| \leq r\}$.

Definition. $\Delta(\alpha)$ is a *Julia line* if for every $\epsilon > 0$

$$n(r, \alpha, \epsilon; 1/(f - a)) \rightarrow \infty \quad (r \rightarrow \infty)$$

for all a with at most two exceptions.

Definition. $\Delta(\alpha)$ is a *Borel line of order $\rho(> 0)$ of the meromorphic function f* if for every $\epsilon > 0$

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, \alpha, \epsilon; 1/(f - a))}{\log r} = \rho$$

for all a with at most two exceptions

Definition. $\Delta(\alpha)$ is a *line of accumulation of f of order ρ* if

$$\limsup_{r \rightarrow \infty} \frac{\log[n(r, \alpha, \epsilon; f) + n(r, \alpha, \epsilon; 1/f)]}{\log r} = \rho.$$

Julia proved that every meromorphic function has a line of Julia. E. Borel proved that every meromorphic function of positive finite order has a Borel line.

Example. $F(z) = \exp(z)$ has the negative and the positive imaginary axes as Julia lines which are also Borel directions.

In general the set of Julia lines will be much more complicated. Functions with $T(r, f) = O(\log^2 r)$ have no Julia lines. On the other hand, J. M. Anderson and J. J. Clunie have shown that, given any closed set E of α on the unit circle, one can construct a meromorphic function f whose set of Julia lines is $\{\Delta(\alpha) | \alpha \in E\}$.

An important modern contribution to the theory of the angular distribution of values of meromorphic functions is the work of Yang Lo and Zhang Guan-Hou, the author of the book under review. This was pioneering work in more ways than one, since the collaboration of the two mathematicians began while they were working under very hard conditions on a pig farm, as "reeducation" decreed by the Cultural Revolution. They literally risked their lives for their research.

Yang and Zhang undertook a very thorough study of the relations between the number of deficient values, the number of asymptotic values, and the configuration of the Borel and Julia lines of functions of finite order. Here is an example.

Theorem. Let $f(z)$ be an entire function, $0 < \lambda < \infty$. Let q be the number of Borel directions of order λ , and let p be the number of finite deficient values of f . Then $p \leq q/2$. Also, if $q < \infty$, $p < 2\lambda$.

Interesting work in Yang-Zhang theory is due to Wu Sheng Jian [W2]:

Let $f(z)$ be a meromorphic function of order λ and lower order μ , $0 \leq \lambda \leq \infty$.

Theorem. Let $\mu < \rho < \lambda$. If f has p ($1 < p < \infty$) deficient values other than 0 and ∞ , then any sector of opening larger than

$$\max \left(\frac{\pi}{\rho}, 2\pi - \frac{\pi}{\rho} \sum_{j=1}^p \arcsin \sqrt{\frac{\delta(a_j, f)}{2}} \right)$$

contains a line of accumulation of order $\geq \rho$.

Theorem. Let $\mu \leq \rho < \lambda$. If $\delta(a, f) > 0$ for an $a \notin \{0, \infty\}$ and if the plane is divided into m ($1 \leq m \leq \infty$) sectors S_j by the lines of accumulation of order $\geq \rho$, then $\lambda \leq \frac{\pi}{\omega}$, where ω is the minimum of the vertex angles of the S_j .

Recently Sodin [S] published a potential theoretic method for the study of the angular distribution of values. This method allows the proof of all the results of Yang-Zhang theory under weaker hypotheses.

Another method is due to Barsegian [B1, B2]. It is based on the Ahlfors theory of covering surfaces, and its results are not immediately comparable to those of the other methods.

8. ASYMPTOTIC VALUES

R. Nevanlinna conjectured that every finite deficient value is asymptotic. This was disproved by counterexamples, first for meromorphic functions (Gol'dberg, 1954) and then for entire functions (Arakelian, 1966). But there are many cases in which the conjecture can be proved. A striking example is the theorem by Drasin quoted above.

9. FURTHER WORK

In 1973 A. Baernstein introduced a powerful new tool, the “Baernstein star function”

$$T^*(re^{i\theta}, f) = \sup_E \frac{1}{2\pi} \int_E \log |f(re^{i\theta})| d\theta,$$

where E ranges over all measurable subsets of Lebesgue measure 2θ of the interval $[0, 2\pi)$.

This function is mentioned in Zhang’s book but is not used. With its help many proofs could have been simplified.

Methodologically the most important innovation due to the Nevanlinnas was the systematic use of potential theory. The recent development of the theory confirms that Nevanlinna Theory can be regarded as a branch of modern potential theory.

Nevanlinna Theory is a special case of the study of holomorphic mappings from one complex manifold to another. Much work is being done on this topic. The first example was Ahlfors’s theory of meromorphic curves (1941). Aside from their intrinsic interest the modern investigations also lead to a deeper understanding of the original Nevanlinna Theory. For example, the number 2 in the defect relation is derived as the Euler characteristic of the sphere in Ahlfors’s theory.

The new theories give an impressive general overview, but so far they have not developed the wealth of detail that is offered by classical Nevanlinna Theory. To claim that the modern theories supercede it is about as reasonable as saying, “I do not read poetry anymore; I only read deconstructionist criticism.” Zhang’s book deals with its topics in a competent, businesslike manner. It will not appeal to a novice, but it contains useful information for the specialist, even if some of it can now be done more simply.

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