

BOOK REVIEW

Controlled Markov processes and viscosity solutions, by W. H. Fleming and H. M. Soner. Applications of Mathematics, volume 25, Springer-Verlag, New York, 1993, xv+428 pp., \$49.95. ISBN 0-387-97927-1

Dynamic programming, introduced under that name by Richard Bellman in 1957 [1], provides a beautifully intuitive approach to problems of dynamic optimization. Consider, for example, a dynamical system represented by the vector differential equation

$$(1) \quad \frac{d}{dt}x(t) = f(x(t), u(t)).$$

The “state” $x(t)$ could be, for instance, the position and velocity of a moving body, where the “input” or “control” $u(t)$ is a forcing term (thrust, throttle position, etc.). An *admissible pair* over some time interval $[0, T]$ is a pair of functions $\{x(t), u(t), t \in [0, T]\}$ such that (1) is satisfied and $x(0) = x$, some given initial state. There might also be control constraints of the form $u(t) \in U(x(t))$ where $x \rightarrow U(x)$ is some set-valued mapping and/or state constraints $x(t) \in D, 0 \leq t \leq T$, where $D \subset \mathbb{R}^n$ is some set. A typical optimal control problem is to choose an admissible pair to minimize a *cost function* of the form

$$(2) \quad J_x(u) = \int_0^T L(x(t), u(t)) dt + \psi(x(T)).$$

Here L might represent a cost for control effort, while ψ penalizes deviation from some desired final state at time T . Since the trajectory $x(\cdot)$ is fixed by (1), once the control $u(\cdot)$ and initial state x are chosen, the problem reduces to choosing the control function $u(\cdot)$, as the notation $J_x(u)$ in (2) indicates. The *value function* $V(t, x)$ for this problem is defined by

$$(3) \quad V(t, x) = \inf_{u(\cdot)} \left\{ \int_t^T L(x(s), u(s)) ds + \psi(x(T)) \right\},$$

where the infimum is taken over all admissible controls on the interval $[t, T]$ with initial condition $x(t) = x$. Of course, $V(T, x) = \psi(x)$. The *principle of dynamic programming* postulates that for any $\delta \in [0, T - t]$ the following equality holds:

$$(4) \quad V(t, x) = \inf_{u(\cdot)} \left\{ \int_t^{t+\delta} L(x(s), u(s)) ds + V(t + \delta, x(t + \delta)) \right\}.$$

Certainly, this “ought” to be true; since, however, one steers from x to $x(t + \delta)$, the lowest cost thereafter is, by definition, $V(t + \delta, x(t + \delta))$. Now, taking δ small and assuming that V is C^1 (i.e., continuously differentiable), we have

$$\begin{aligned} V(t + \delta, x(t + \delta)) &= V(t, x) + \frac{\partial V}{\partial t} \delta + \frac{\partial V}{\partial x} \frac{dx}{dt} \delta + o(\delta) \\ &= V(t, x) + \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u) \right) \delta + o(\delta). \end{aligned}$$

Substituting this in (3), dividing by δ , and letting $\delta \rightarrow 0$, we obtain the *Bellman equation*

$$(5) \quad \frac{\partial V}{\partial t} = \inf_{u \in U(x)} \left\{ \frac{\partial V}{\partial x}(t, x) f(x, u) + L(x, u) \right\} = 0,$$

$$(6) \quad V(T, x) = \psi(x).$$

It is not necessary to believe in the above heuristics to convince oneself that (5), (6) is the appropriate equation at which to look. Indeed, suppose that W is a C^1 solution of (5), (6), and let $(x(\cdot), u(\cdot))$ be any admissible pair with $x(0) = x$. From (5)

$$(7) \quad \frac{d}{dt} W(t, x(t)) = \frac{dW}{dt} + \frac{\partial W}{\partial t} f(x(t), u(t)) \geq -L(x(t), u(t)).$$

Integrating from 0 to T and using (6), we obtain

$$(8) \quad W(0, x) \leq J_x(u).$$

Now suppose there is a function $u^*(t, x)$ such that

$$(9) \quad \begin{aligned} &\inf_{u \in U(x)} \left\{ \frac{\partial V}{\partial t}(t, x) f(x, u) + L(x, u) \right\} \\ &= \frac{\partial V}{\partial x}(t, x) f(x, u^*(t, x)) + L(x, u^*(t, x)) \end{aligned}$$

and that the equation

$$\frac{d}{dt} x^*(t) = f(x^*(t), u^*(t, x^*(t))), \quad x(0) = x,$$

has a unique solution $x^*(\cdot)$ satisfying the state space constraints (if any). Then taking $\hat{u}(t) := u^*(t, x^*(t))$, the same argument as above goes through with “=” replacing “ \geq ” in (7) to give

$$(10) \quad W(0, x) = J_x(\hat{u}).$$

Now (8), (10) say that \hat{u} is optimal and hence that $W(0, x) = V(0, x)$. An analogous argument shows that $W(t, x) = V(t, x)$, $t \in [0, T]$.

The advantage of the Bellman equation (5), (6) is of course that the original minimization over *functions* $u(\cdot)$ is replaced by a *pointwise minimization* over the control set $U(x)$. Computation of the minimal cost is reduced to solving a first-order nonlinear PDE, and the optimal control \hat{u} is then obtained as in (9). Optimal controls can actually be computed this way in special cases such as the “Linear/Quadratic problem” (see [2] or Example I.5.1 of the book under review).

The problem with this approach is that the value function V may as well fail to be C^1 and therefore cannot satisfy (5), (6) in any classical sense. Fleming and Soner give an example of this, Example II.2.1. This refers to a variant of the above problem in which $x(t)$ is stopped at the first exit time τ from a region O if this occurs before the terminal time T . Thus the cost is

$$J_x(u) = \int_0^{\tau \wedge T} L(x(t), u(t)) ds + \Psi(\tau \wedge T, x(\tau \wedge T))$$

where Ψ is a cost function defined on $\partial^*Q := ([0, T] \times \partial O) \cup (\{T\} \times O)$ and $\tau \wedge T := \min\{\tau, T\}$. In the example, $O = (-1, 1)$, $\Psi = 0$, $dx/dt = u \in \mathbb{R}$, $T = 1$, and $L(x, u) = 1 + u^2/4$. Then it turns out that

$$(11) \quad V(t, x) = \begin{cases} 1 - |x|, & |x| \geq t, \\ 1 - t, & |x| \leq t, \end{cases}$$

which is not differentiable at $t = |x|$. The formal Bellman equation is

$$(12) \quad 0 = \frac{\partial V}{\partial t} + \min_u \left(\frac{\partial V}{\partial x} u + 1 + \frac{1}{4} u^2 \right) = \frac{\partial V}{\partial t} - \left(\frac{\partial V}{\partial x} \right)^2 + 1$$

which is satisfied by (11) except at $|x| = t$. In view of this, an obvious expedient is to look for functions which are Lipschitz continuous and satisfy the Bellman equation almost everywhere. However, this does not work because, as the authors show, there are infinitely many different such functions all satisfying (12) and the appropriate boundary condition.

In the preface to [3], Pontryagin et al. hold up dynamic programming like a dead rat by its tail: “It must be noted that the assumption on the continuous differentiability of the [value] functional does not hold in the simplest cases. Thus, Bellman’s considerations yield a good heuristic method rather than a mathematical solution of the problem.” For this reason and because of the prestige of the Soviet school, “Bellman’s considerations” were, for a period of twenty years or so, almost entirely superseded in deterministic optimal control theory by methods based on the famous Maximum Principle [3] which, as is well known, gives a *necessary* condition for optimality by methods analogous to those of the calculus of variations. The Maximum Principle is valid under much weaker conditions than those required for validity of the dynamic programming method, and computational methods for finding points satisfying the necessary conditions have been studied [4], although this subject is surprisingly problematic in that numerical methods necessarily involve discretization and there are no satisfactory counterparts to the maximum principle for discrete optimal control problems.

In continuous-time *stochastic* control, the situation is very different, for a simply stated technical reason. The best-developed part of this subject concerns the diffusion model, in which the dynamic equation (1) is replaced by the Itô stochastic differential equation

$$dx(t) = f(x(t), u(t))dt + \sigma(x(t)) dw(t).$$

Here $w(t)$ is a vector Brownian motion process, and to make sense of the equation, the control process $u(t)$ has to be at least “nonanticipative”; i.e., for any times $r \leq s \leq t$, $w(r)$ is independent of the increment $w(t) - w(s)$. The value function is now defined as the minimal *expected* cost

$$V(t, x) = \inf_{u(\cdot)} E_{t,x} \left\{ \int_t^T L(x(s), u(s)) ds + \psi(x(T)) \right\},$$

where the subscript (t, x) indicates the initial condition $x(t) = x$. Formal arguments analogous to those used above show that the Bellman equation “morally” satisfied by V in this case is

$$(13) \quad \frac{\partial V}{\partial t} + \min_{u \in U(x)} \left\{ \frac{\partial V}{\partial x} f(x, u) + \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2 V}{\partial x^i \partial x^j} + L(x, u) \right\} = 0$$

with the boundary condition $V(T, x) = \psi(x)$. Here $a = \sigma \sigma^T$. If the process is stopped on exit from a set O , with boundary cost function $\Psi(t, x)$, then the boundary condition is simply

$$(14) \quad V(t, x) = \Psi(t, x), \quad (t, x) \in \partial^* Q,$$

where $\partial^* Q$ is as defined above. In (13) the second derivatives of V arise from the rules of Itô calculus which, roughly speaking, state that $dw^i dw^j \simeq I_{(i=j)} dt$, mandating the retention of second-order terms in the Taylor expansion.

If the problem is “uniformly elliptic”—i.e., $\sigma \sigma^T(x) \geq \delta I$, where I denotes the identity matrix and $\delta > 0$, meaning that $\sigma \sigma^T$ is uniformly positive definite—then (13) has a classical ($= C^{1,2} = C^1$ in t , C^2 in x) solution under much weaker conditions than its deterministic counterpart, and dynamic programming is therefore a viable basis for a theory of stochastic control; it is described in Professor Fleming’s previous book [5] with R. W. Rishel. The intuition is clear: under the uniform ellipticity condition the controlled process $x(t)$ is similar to Brownian motion in that, starting at a fixed point x at time 0, it has a smooth density function of unbounded support at any time $t > 0$, and the sample path $t \mapsto x(t)$ is continuous but highly irregular. In particular, whatever the control, the process adamantly refuses to stay with probability 1 on one side of any barrier in the state space for any positive interval of time, so there cannot be any discontinuous or violent changes in the value function $V(t, x)$ as x is varied. In David Williams’s memorable phrase, “The rougher the sample paths, the smoother the analysis.”

An additional reason why dynamic programming has proved a superior method in stochastic control is that the stochastic analogue of the Pontryagin Maximum principle is not very useful, at least from the computational standpoint. This is because the “adjoint variable” involves a conditional expectation over future sample paths, so computing it is equivalent to computing a function-space integral, a daunting task. If the uniform ellipticity condition $\sigma \sigma^T \geq \delta I > 0$ is not satisfied, then in general stochastic problems are more like deterministic ones in that smoothness of the value function cannot be guaranteed (although sometimes “hypoellipticity” saves the day), leaving us in need of some reasonable interpretation for (13).

Continuous-time dynamic programming is far too good an idea to be written off just because the Bellman equation may not make sense as it stands. There have been two major efforts at providing “weak solutions” to the Bellman equation: one based on “nonsmooth analysis” initiated by F. H. Clarke [6] and the theory of “viscosity solutions”, introduced by Crandall and Lions [7], which is the subject of the book under review. The idea of viscosity solutions is easily stated. Write the stochastic Bellman equation (13) as

$$(15) \quad -\frac{\partial V}{\partial t} + F(x, DV, D^2V) = 0$$

where DV and D^2V denote the vector or matrix of first and second derivatives in x respectively and

$$F(x, p, q) = \max_{u \in U(x)} \left\{ -p \cdot f(x, u) - \frac{1}{2} \sum_{i,j} q_{ij} a_{ij}(x) - L(x, u) \right\}.$$

Note that F satisfies the monotonicity property

$$(16) \quad F(x, p, q + q') \leq F(x, p, q)$$

for any symmetric matrices q, q' with $q' \geq 0$. Now suppose that $v, V \in C^{1,2}$ are such that, for some (t_0, x_0) ,

$$(17) \quad v(t, x) \geq V(t, x) \quad \text{for all } (t, x) \quad \text{and} \quad v(t_0, x_0) = V(t_0, x_0).$$

Then by elementary calculus we must have $\partial V/\partial t = \partial v/\partial t$, $DV = Dv$, and $D^2v - D^2V \geq 0$. Thus if V satisfies (15), then using (16) we have

$$(18) \quad -\frac{\partial v}{\partial t}(t_0, x_0) + F(x_0, Dv(t_0, x_0), D^2v(t_0, x_0)) \leq 0.$$

We say that a continuous function V is a *viscosity subsolution* of (15) if, for any (t_0, x_0) , (18) holds for all $v \in C^{1,2}$ satisfying (17). Since only derivatives of v are involved in (18), these conditions are equivalent to requiring that $v \in C^{1,2}$ and $V - v$ has a local maximum at (t_0, x_0) . Similarly, V is a *viscosity supersolution* of (15) if it is continuous and

$$-\frac{\partial v}{\partial t}(t_0, x_0) + F(x_0, Dv(t_0, x_0), D^2v(t_0, x_0)) \geq 0$$

for every $v \in C^{1,2}$ such that $V - v$ has a local minimum at (t_0, x_0) . V is a *viscosity solution* of (15) if it is both a subsolution and a supersolution. Clearly then, any classical ($C^{1,2}$) solution is a viscosity solution, but the viscosity solution is well defined for functions not necessarily possessing the requisite derivatives. A viscosity solution of the deterministic Bellman equation (5) is simply the special case of (15) with $a_{ij} \equiv 0$. Some care is required over boundary conditions, and the authors introduce the notion of *constrained viscosity solutions* to cover deterministic problems with state space constraint $x(t) \in \bar{O}$, $0 \leq t \leq T$. Roughly speaking, a

constrained viscosity solution is a subsolution on $[0, T[\times O$ and a supersolution on $[0, T[\times \overline{O}$.

An attractive feature of these definitions is that value functions of control problems are easily shown to be viscosity solutions of the corresponding Bellman equation. Take for example the deterministic problem and *define* V by (3). Now suppose that $v \in C^{1,2}$, $v \geq V$, and $v(t_0, x_0) = V(t_0, x_0)$. Then from (4)

$$(19) \quad v(t_0, x_0) \leq \inf_{u(\cdot)} \left\{ \int_{t_0}^{t_0+\delta} L(x(s), u(s)) ds + v(t_0 + \delta, x(t_0 + \delta)) \right\}.$$

Now doing a Taylor expansion of v and following the steps that led from (4) to (5), we immediately conclude from (19) that V is a viscosity subsolution of (5). If there exists an optimal control \hat{u} from any starting point (t_0, x_0) , then an equally simple argument shows that V is a viscosity supersolution; this argument is little more complicated when only ε -optimal controls exist. Thus there is a natural symbiosis between problems of optimal control and viscosity solutions. The method is not, however, restricted to Bellman equations, and viscosity solutions can be defined for wide classes of nonlinear PDEs taking the form (15) provided that the function F satisfies (16). On the other hand, there is no ready extension to *systems* of PDEs, since order properties appear in an essential way in the definition.

One problem with viscosity solutions, at least from the outsider's point of view, is that *too much* has been written about them. Since the original research announcement in 1981 [8], several hundred publications have appeared on the subject, giving a plethora of subtly varying definitions, general results and applications. These include lengthy survey articles [9], a Lecture Notes volume [10], and even a videotape [11], but the present volume is to be welcomed as apparently the first systematic treatise on the subject. As the title indicates, the coverage is centered almost entirely around topics in control theory and stochastic analysis. Chapters on deterministic and stochastic optimal control are followed, respectively, by chapters on viscosity solutions for first- and second-order PDEs, while the concluding chapter deals with discretization and convergence of numerical schemes. There are two further chapters, discussed below, which are less centrally connected with viscosity solutions and which deal with special topics in stochastic analysis and control.

A novel feature is that the authors unify the presentation of viscosity solutions by axiomatising the properties of an abstract nonlinear semigroup \mathfrak{J}_{st} . As an example, consider the deterministic optimal control problem (1), (2) and write the value function V of (3) as $V(t, x) = \mathfrak{J}_{tT}\psi(x)$. Then the dynamic programming principle (4) is equivalent to the statement that \mathfrak{J}_{st} is a semigroup, i.e., $\mathfrak{J}_{tT}\psi = \mathfrak{J}_{t,t+\delta}(\mathfrak{J}_{t+\delta,T}\psi)$, and the Bellman equation (5) is equivalent to a statement involving the generator of \mathfrak{J}_{st} . A similar approach can be taken to stochastic control, differential games, etc.; and by taking the semigroup as primitive, the authors achieve a more unified presentation. Only nonlinear PDEs will be discussed in the remainder of this review, but it should be borne in mind that some results apply more generally.

The heart of viscosity solution theory lies in the uniqueness theorems. The approach taken to uniqueness is as follows: suppose W and V are, respectively, a viscosity sub- and supersolution of, say, (15) in a region $Q = [0, T] \times O$ and that if O is unbounded, W and V are bounded and uniformly continuous (BUC) on \overline{Q} . It is then shown that

$$(20) \quad \sup_{\overline{Q}}(W - V) = \sup_{\partial^* Q}(W - V).$$

This implies that there is at most one viscosity solution of (16) satisfying the boundary condition (15): If Y and Z are viscosity solutions, then each is both a subsolution and a supersolution; so taking $W = Y$ and $V = Z$ in (20), we conclude that $Y - Z \leq 0$, and interchanging Y and Z gives $Z - Y \leq 0$. The methods used to prove (20) are quite different for first- and second-order equations, though both rely on contradiction arguments. The first-order case is much the simpler, and the proof is based on taking quadratic functions for v (see the definition of viscosity solution above). This approach fails in the second-order case, and the authors give an argument based on *Jensen's maximum principle* for semiconvex functions. A real-valued continuous function f on a compact set $G \subset \mathbb{R}^d$ is semiconvex if $x \mapsto f(x) + K|x|^2$ is convex for some $K \geq 0$. At its maxima a semiconvex function f is differentiable but not necessarily twice differentiable; Jensen's maximum principle states that f is twice differentiable at points arbitrarily close to the maxima and that at such points the Hessian is nonnegative and the gradient small. Using this result, (20) is readily established when W and V are semiconvex and semiconcave respectively. General W, V can be approximated by semiconcave [-concave] functions using "inf" and "sup" convolutions. In both the first- and second-order cases the proofs are given for bounded time-space set Q or for bounded viscosity solutions, ruling out many problems with unbounded cost functions, though some extensions to polynomially growing solutions are mentioned for the first-order case. Further extensions are given by Crandall, Ishii, and Lions [9]. These uniqueness results are the major achievements of the theory.

Another aspect of the theory with important ramifications is *stability*, i.e., the behaviour of solutions under perturbation of the coefficients. Suppose that F^ε is a family of continuous functions satisfying (16) and that for each $\varepsilon > 0$, V^ε is a viscosity solution of (15) with F^ε replacing F . Then it is simple to show that if $F^\varepsilon(x, p, q) \rightarrow F(x, p, q)$ and $V^\varepsilon(t, x) \rightarrow V(t, x)$ uniformly on compact sets as $\varepsilon \rightarrow 0$, then V is a viscosity solution of (15). The key point here is that no convergence of derivatives is required. The prototype application is to take

$$F^\varepsilon(x, p, q) = -\frac{1}{2}\varepsilon \sum_i q_{ii}^2 + F(x, p).$$

The equation

$$-\frac{\partial V^\varepsilon}{\partial t} + F^\varepsilon(x, DV^\varepsilon, D^2V^\varepsilon) = 0$$

is then uniformly elliptic and typically has a classical solution (which of course coincides with the viscosity solution). If $V^\varepsilon \rightarrow V$ uniformly on compacts, then V is the unique viscosity solution of the first-order equation

$$(21) \quad -\frac{\partial V}{\partial t} + F(x, DV) = 0.$$

This approach to solving (21) is sometimes known as the "method of vanishing viscosity", hence the term "viscosity solutions" (which actually seems rather poorly chosen, though obviously it is far too late for a change).

To use singular perturbation methods in an effective way, one needs to establish the existence of convergent subsequences V^{ε_n} . One approach is to establish precompactness of $\{V^\varepsilon : \varepsilon > 0\}$, possibly by obtaining a uniform Hölder estimate

and appealing to the Arzelà-Ascoli theorem. There is, however, a more powerful method, due to Barles and Perthame [12], involving only a uniform boundedness condition. The details are complicated and cannot be given here; they involve the introduction of a possibly discontinuous viscosity solution. The procedure is discussed in Chapter VII of the book, and two applications are given: to “vanishing viscosity” and to a large deviation problem for exit probabilities of a small-noise diffusion from the cylinder $[0, T] \times O$. The latter is described below.

A closely related question is that of numerical methods. There would be little point in showing that some function of interest is the unique viscosity solution of a nonlinear PDE if there were no way of computing, at least in principle, this solution. It is one of the most satisfying aspects of the theory that natural discretization schemes converge to a viscosity solution under very mild conditions. Together with the uniqueness theorems, such results guarantee that in applying standard numerical schemes, one is computing the right thing. These results are surprisingly recent and are mainly due to Barles and Souganidis [13]; they are discussed in the final chapter of the book. To solve, say, (15) with boundary condition $V(T, x) = \psi(x)$, one introduces a discretization parameter $h > 0$ and an abstract finite-difference equation of the form

$$(22) \quad V^h(t, x) = F_h[V^h(t + h, \cdot)](x)$$

for $t = t_0 + kh$, $k = 0, 1, \dots$, with the boundary condition

$$(23) \quad V^h(T, x) = \psi(x).$$

The requirements on the family of functions F_h are (a) monotonicity; (b) $F_h(\phi + c) = F_h(\phi) + c$ for all $c \in \mathbb{R}$; (c) stability (i.e., existence of uniformly bounded solutions of (22), (23) for $0 < h < 1$); and, most importantly, (d) consistency, i.e., for $w \in C^{1,2}$

$$\lim_{\substack{(s,y) \rightarrow (t,x) \\ h \downarrow 0}} \frac{1}{h} \{F_h[w(s+h, \cdot)](y) - w(s, y)\} = w_t - F(x, Dw, D^2w).$$

Under these conditions $V^h \rightarrow V$ uniformly on compact sets, where V is a viscosity solution of (15) with $V(T, x) = \psi(x)$. When there are lateral boundary conditions as in (14), a little more care is required, but the result is basically the same. The “abstract” finite-difference scheme (22) is realized in practice by using the discretization procedure pioneered by H. J. Kushner and described in detail in [14]. In this procedure derivatives are replaced by approximating finite-difference quotients in an ingenious way so that (22) becomes the Bellman dynamic programming equation for a controlled discrete Markov chain which approximates the controlled diffusion. Thus in solving (22), (23), one is computing an exact solution to an approximate problem as well as an approximate solution to the exact problem, and this enhances the numerical properties of the solution as well as the credibility of the results for finite h . In Kushner’s treatment, convergence is proved by methods based on weak convergence of probability measures. The viscosity solution approach provides an alternative which is convenient in some cases; the reviewer and collaborators certainly found it so in an option-pricing problem of mathematical finance [15].

As mentioned above, the book contains, in addition to the material just sketched out, two chapters on topics where viscosity solutions intervene in a less central way but where the authors have made important contributions. The first of these (Chapter VI) concerns logarithmic transformations. The idea is as follows: suppose that $\Phi(t, x)$ is a *positive* solution of a linear PDE of the form

$$(24) \quad \frac{\partial \Phi}{\partial t} + \mathfrak{A}\Phi + q(x)\Phi = 0$$

where \mathfrak{A} is the differential generator of a Markov diffusion process, i.e.,

$$(25) \quad \mathfrak{A}f = b(x)\nabla f + \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Now define $V = -\log \Phi$; then, as is easily checked, V satisfies

$$(26) \quad -\frac{\partial V}{\partial t} - \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2 V}{\partial x^i \partial x^j} + H(x, D_x V) = 0$$

where

$$\begin{aligned} H(x, p) &= -b(x)^T p + \frac{1}{2} p^T a(x) p + q(x) \\ &= \max_{u \in \mathbb{R}^d} \left\{ -p^T u - \frac{1}{2} (b(x) - u)^T a^{-1}(x) (b(x) - u) - q(x) \right\}. \end{aligned}$$

Thus (26) is the Bellman equation corresponding to the stochastic control problem of minimizing

$$E \int_0^\tau \frac{1}{2} (b(x_t) - u_t)^T a^{-1}(x_t) (b(x_t) - u_t) dt + \Psi(\tau, x_\tau)$$

for the controlled diffusion (x_t) given by

$$dx_t = u_t dt + \sigma(x_t) dw_t,$$

where $a = \sigma\sigma^T$, τ is the exit time from $[0, T] \times O$, and $\exp(-\Psi)$ is the boundary data for (24). This transformation has various applications, the principal one being to the Wentzell-Freidlin large deviations theorem for exit probabilities. If we take $q = 0$ in (24), replace \mathfrak{A} by \mathfrak{A}^ε , where \mathfrak{A}^ε is as in (25) but with a_{ij} replaced by εa_{ij} , and take as boundary data $\Psi(t, x) = 1$, $(t, x) \in [0, T[\times \partial O$, $\Psi(T, x) = 0$, $x \in O$, then $\Phi(t, x) = E_{tx}[\theta^\varepsilon < t_1]$ where θ^ε is the first exit time from O for the diffusion ζ_t^ε given by

$$d\zeta_s^\varepsilon = b(\zeta_s^\varepsilon) ds + \sqrt{\varepsilon} \sigma(\zeta_s^\varepsilon) dw_s, \quad \zeta_t^\varepsilon = x.$$

Denote $V^\varepsilon(t, x) = -\varepsilon \log \Phi(t, x)$. Then as $\varepsilon \rightarrow 0$, $V^\varepsilon \rightarrow V^0$ satisfying

$$(27) \quad -\frac{\partial V^0}{\partial t} + H(x, D_x V^0) = 0$$

with the boundary condition $V^0(t, x) = 0$, $(t, x) \in [0, T] \times \partial O$, $V^0(T, x) = \infty$, which is the Bellman equation corresponding to the calculus of variations problem

of minimizing $\int_t^T L(\zeta(s), \dot{\zeta}(s)) ds$ over all absolutely continuous curves $\zeta(\cdot)$ such that $\zeta(\tau) \in \partial O$, the latter requirement being enforced by the boundary condition. Here $L(\zeta, p) = \frac{1}{2}(b(\zeta) - p)^T a^{-1}(\zeta)(b(\zeta) - p)$. This shows that, roughly speaking, the small-noise exit probability $\Phi^\varepsilon = \exp(-V^\varepsilon/\varepsilon)$ satisfies $\Phi^\varepsilon \sim \exp(-\frac{1}{\varepsilon}V^0)$. It is also possible in certain cases to get a ‘‘WKB’’ expansion of the form

$$\Phi^\varepsilon = \exp \left\{ -\frac{1}{\varepsilon} V^0 [W_0 + \varepsilon W_1 + \cdots + \varepsilon^m W_m] + o(\varepsilon^m) \right\}.$$

The first statement is expressed rigorously as

$$\Phi^\varepsilon(t, x) = \exp \left\{ -\frac{1}{\varepsilon} [V^0(t, x) + h^\varepsilon(t, x)] \right\},$$

where $h^\varepsilon \rightarrow 0$ uniformly on compact sets and V^0 is the unique viscosity solution of (27), by using the stability properties of viscosity solutions.

The other chapter in which viscosity solutions play more of a supporting role is Chapter VIII, dealing with problems of so-called singular stochastic control. The simplest example of this is as follows: suppose w_t is a scalar Brownian motion and we wish to minimize

$$(28) \quad E \left\{ \int_0^\infty e^{-\delta t} x_t^2 dt + \int_0^\infty e^{-\delta t} d\check{\xi}_t \right\}$$

where $\delta > 0$, $x \in \mathbb{R}$ and $x_t = x + w_t + \xi_t$, and (ξ_t) is an adapted *bounded variation* process, with $\check{\xi}_t$ denoting the total variation on $[0, t]$. The interpretation, roughly, is that we are attempting to ‘‘steer’’ (x_t) to zero with minimum ‘‘use of fuel’’. It turns out that the optimally controlled process (x_t) is Brownian motion reflected at barriers $\pm b$, so that $\xi_t = L_t - U_t$ where $(L_t), (U_t)$ are the local times at $-b, +b$ respectively. If $x > b$ [$x < -b$], there is an initial jump to b [$-b$]. (ξ_t) has sample paths which are continuous but singular with respect to Lebesgue measure; hence the term ‘‘singular control’’. The barrier b is determined by solving the variational inequality

$$(29) \quad \min \left\{ \frac{1}{2} \frac{d^2 V}{dx^2} - \delta V + x^2, \frac{dV}{dx} - 1, \frac{dV}{dx} + 1 \right\} = 0.$$

This has a unique symmetric globally C^2 solution $V(x)$ and $[-b, b] = \{x : \frac{1}{2}d^2V/dx^2 - \delta V + x^2 = 0\}$; $V(x)$ is the minimal cost starting at x . (29) can be derived formally by writing down the Bellman equation for (28) with absolutely continuous control $d\xi = udt$ satisfying $|u| \leq \kappa$ and then letting $\kappa \rightarrow \infty$.

Many special problems of this kind have been solved in the last fifteen years or so, most of them one dimensional: equation (29) is a free boundary problem, and construction of the appropriate reflecting diffusions cannot be carried out in dimension $n > 1$ unless the boundary is known to be sufficiently smooth. It is possible under reasonable conditions to show, as the authors do, that the value function of a multidimensional singular control problem is a viscosity solution of the appropriate analog of (29), but in a way results of this kind do not show viscosity solution theory in its best light: all the really interesting questions, i.e.,

what the free boundary looks like and how the optimal processes (if they exist) are constructed, are swept under the carpet. In this area it is undoubtedly true that most of the vitality still comes from a detailed examination of special cases rather than any general theory, and here viscosity solutions have less of a role to play, although they have been effectively used as an intermediate stage in showing that variational inequalities similar to (29) actually have classical (C^2) solutions.

In summary, viscosity solutions are no passing fashion; they are here to stay, adding another chapter to the fascinating saga of the interplay between stochastic analysis and partial differential equations. Of course, they do not solve every problem: they have nothing to say about existence of optimal controls or necessary conditions, for which one should use nonsmooth analysis [6]; and the computational methods are limited, like all numerical solution of PDEs, to problems of low spatial dimension. They do, however, provide an effective approach to many problems, control-related and otherwise, involving nonlinear PDEs for which more traditional methods cannot be used. Fleming and Soner's book contains an enormous amount of information, much more than summarized here. It is written in an uncompromising, occasionally even terse, style, but there are helpful introductory paragraphs at the beginning of most sections explaining where the argument is going. This is an indispensable reference for researchers in control theory, and anyone with interests in nonlinear PDEs or applications of stochastic analysis will find something useful in it too.

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