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Word processing in groups, by David B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. Jones & Bartlett, 1992, xi + 330 pp., \$59.00. ISBN 0-86720-244-0

Let G be a group, let X be a subset of G, and let X^{-1} be the set of inverses of the elements of X. Then one says that X *generates* G if every element of G can be expressed as a product of the elements in $L = X \cup X^{-1}$. A finite string of elements of L is termed an X-*word*; such an X-word is termed a *relator* if it takes on the value 1 in G, i.e., the result of multiplying its entries is the identity element of G. One writes

$$G = \langle X ; R\rangle$$

if G is generated by X, if R is a set of relators, and if everything about G can be deduced from this data and the fact that G is a group. Such a description is termed a *presentation* of G; it is termed a *recursive presentation* if X is a, possibly infinite, countable set

$$X = \{x_1, x_2, \dots\}$$

and R is a recursively enumerable subset of the set of all X-words; and it is termed a *finite presentation* if both X and R are finite. A group is termed *recursively presentable* if it has a recursive presentation; and it is termed *finitely presentable*, or, as is more usual, *finitely presented*, if it has a finite presentation. A recursive presentation $\langle X ; R\rangle$ (of a group G) is said to have a *solvable word problem* if the set of all relators is a recursive subset of the set of all X-words. It is possible for one recursive presentation of a group G to have a solvable word problem and for another such recursive presentation not to have a solvable word problem. However, all finite presentations of a finitely presented group G either have a solvable word problem or none of them do; we express this property of a finitely presented group G by simply saying that G has a solvable word problem or an unsolvable word problem, as the case may be.

Finitely presented groups arise in many branches of mathematics, in particular in algebraic topology, and have been the subject of intensive study. One of the primary concerns in this study has been the search for computable invariants. In 1954 P. S. Novikov constructed a finitely presented group G with an *unsolvable word problem*; thus, if $\langle X ; R\rangle$ is a finite presentation of G, there is no algorithm which determines whether or not any X-word takes the value 1 in the group G. The existence of a finitely presented group with an unsolvable word problem led to a host of negative algorithmic results about finitely presented groups. Indeed, it is now known that almost all problems about finitely presented groups, their subgroups, and their elements are algorithmically undecidable. For example, Novikov's student Adian proved that there is no algorithm which decides whether or not any finitely presented group is of order one (the triviality problem), is abelian, is finite, is a group of matrices, and so on.

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It follows that there is no computable set of invariants which determines any finitely presented group (up to isomorphism).

It turns out that these negative algorithmic results about finitely presented groups are not isolated phenomena but accurately reflect their complexity. For, in 1961, G. Higman proved the following remarkable theorem:

A finitely generated group is isomorphic to a subgroup of a finitely presented group if and only if it is recursively presentable.

Although it is not apparent from the formulation above, it follows from Higman's theorem that the complexity of any recursive function can be encoded into the relators of a finitely presented group. This, for instance, leads to the existence of a finitely presented group which contains an isomorphic copy of every finitely presented group!

Despite the algorithmic intractability of finitely presented groups, there are many classes of finitely presented groups that appear to be amenable to systematic study. The extraordinary developments in the study of fundamental groups of three-dimensional manifolds attest to this assertion.

In the book under review, the authors introduce and study a strikingly new class of groups, which they term *automatic groups*, opening up a completely new world of infinite groups for exploration. Their very definition allows for the actual computation of the product of two elements in a way that is akin to, but simpler than, the multiplication of two integral matrices. This conjures up the fascinating possibility that some of the exploration of these automatic groups can be carried out by means of high-speed computers.

The actual definition of an automatic group depends on the notion of a *finite-state automaton.* Such finite-state automata and slight generalisations of them provide the underpinnings of today's remarkably efficient word processors and compilers. It suffices here, taking the notion of a finite-state automaton for granted for the moment, to define an automatic group, a little imprecisely, as follows. A group G, equipped with a finite set

$$L = \{a_1, \dots, a_n\}$$

of generators, which we assume for convenience to be closed under inversion, is termed *automatic* if there exist $n+2$ finite-state automata M, $M_=$, M_i $(i = 1, \dots, n)$, such that:

(1) the input of M consists of all X-words, and its output is a set $L(M)$ of X-words whose values comprise all of G;

(2) the input of the so-called equality checker $M_=$ is $\{(u, v) \mid u, v \in L(M)\}$, and the output of $M_=$ consists of $\{(u, v) \mid u, v \in L(M),\ u =_G v\}$, where the notation $u =_G v$ expresses the fact that the words u and v are equal in G;

(3) the input of the so-called multiplication checker M_i is the set $\{(u, v) \mid u, v \in L(M)\}$, and the output of M_i consists of $\{(u, v) \mid u, v \in L(M),\ u =_G va_i\}$.

In other words *a group is automatic if there is a machine* (*i.e., a finite-state automaton*) *which provides a list of words representing its elements, a machine which checks for equality, and a bunch of machines which check whether words represent elements which differ by right multiplication by a generator.* The existence of such machines turns out to be independent of the choice of the finite set L of generators of G. These machines, together with a set of generators of an automatic group G, is termed *an automatic structure* for G.

It remains only to define a finite-state automaton. To this end, let L be a finite set and let $L^\star$ be the set of all *words*

$$w = a_1 \cdots a_n \qquad (a_i \text{ in } L)$$

(including the empty word e), where each letter $a_i \in L$. The subsets of $L^\star$ are often referred to as *languages over* L. Then a *finite-state automaton* is a quintuple

$$M = (S, Y, L, \tau, s_0),$$

where

(1) S is a finite set of *states*;
(2) Y is a subset of S, the set of *yes states* or *accept states* or *final states*;
(3) L is a finite set of *letters*;
(4) τ is a function from $S \times L \to S$, *the transition function*;
(5) s_0 is an element of S, the *initial state* or *start state.*

M can be thought of as a machine with a head that scans a tape, which is in a vertical position and is fed into M. The tape is divided into a finite number of squares. Each square has a letter printed on it. The top of the tape is fed into M, which starts up in the initial state s_0. M reads the first letter on the tape, the tape moves up so that the machine now scans the second letter on the tape, whereupon the machine goes into a new state. This new state is determined by the transition function τ and the first letter scanned by M. The process continues with each new state being determined by the preceding state and the letter that was scanned while the machine was in that state. When the last letter on the tape is read, the machine goes into a new state and stops. If the last state is an accept state, then the string of letters on the tape is *accepted* by M; otherwise it is *rejected*. The *language of* M is the set $L(M)$ of words accepted by M. The language $L(M)$ of M can therefore be described as follows. Given a word $w = a_1 \cdots a_n$ $(a_i \in L)$, let $t_0 = s_0$, and let $t_i = \tau(t_{i-1}, a_i)$ $(i = 1, \ldots, n)$. Then

$$L(M) = \{w = a_1 \cdots a_n \mid t_n \in Y\}.$$

The languages recognized by finite-state automata are called *regular languages.* These languages provide the basis for the definition of an automatic group.

The origins of automatic groups can be traced back to the work of Max Dehn and, more importantly, to the recent work of Cannon. Cannon's work, which is geometric in nature, can also be regarded as a precursor to work of Gromov, in which he introduces and creates the theory of what he terms *word-hyperbolic groups.* In contrast to automatic groups, the definition of word-hyperbolic groups has a geometric flavor, and the groups lend themselves to geometric analysis. Hyperbolic groups, like automatic groups, have generated a great deal of activity and interest, revolutionising part of combinatorial group theory. The relevance of word-hyperbolic groups to this discussion is due to the fact that *word-hyperbolic groups are automatic.* They are probably considered, by topologists, to be the most important class of automatic groups. One way of defining word-hyperbolic groups is by means of the *Cayley graph,* which also plays an important role in the study of automatic groups. The Cayley graph $\Gamma = \Gamma(G) = \Gamma_X(G)$ of a group G, relative to a set X of generators of G, is a directed graph with vertex set G and edges all triples (g, a, ga), where $g \in G$, $a \in L = X \cup X^{-1}$. The element g is termed the *origin*, h the *terminus*, and a the *label* of the edge (g, a, h). The edges (g, a, ga) and (ga, a^{-1}, g) are referred to as *inverses*, and the origin and terminus of an edge are referred to as its *extremities*. A sequence γ of (not necessarily distinct) vertices $g_0, \ldots, g_n$ of Γ is termed a *path of length* n if either $n = 0$ or, in the case where $n > 0$, if for each $i = 0, \ldots, n-1$ either $g_i = g_{i+1}$ or there exists an edge whose origin is g_i and whose terminus is g_{i+1}. The element g_0 is called the *origin* and g_n the *terminus* of γ, g_0 and g_n are referred to as the *extremities* of γ, and one says that γ *goes* from g_0 to g_n. An infinite sequence $g_0, \ldots$ of vertices such that for each $i \geq 0$, either $g_i = g_{i+1}$ or there is a directed edge whose origin is g_i and whose terminus is g_{i+1} is also sometimes referred to as a *path*. As above, g_0 is the *origin* of such a path. Since L is closed under inverses, for each path that goes from g to h there is a corresponding path that goes from h to g. Since L consists of a set of generators of G together with their inverses , it follows that the Cayley graph is connected. If g and h are vertices in a Cayley graph Γ, then the distance $d(g, h)$ between them is defined to be the minimum length of a path from g to h. This turns Γ into a metric space. The vertices of Γ, i.e., the elements of G, are sometimes referred to as *points*. A shortest path from g to h is termed a *geodesic*, and a *triangle* in Γ is termed a *geodesic triangle* if its sides are geodesics. Gromov has defined a geodesic triangle to be *δ-thin* if every point on one side of the triangle is no further than δ from at least one point on one of the other two sides, i.e., each side of the triangle is contained in a δ-neighbourhood of the union of the other two sides. The group G is then termed *word-hyperbolic* if there exists a δ such that every geodesic triangle in Γ is δ-thin. These word-hyperbolic groups can be described rather differently by means of so-called *isoperimetric inequalities.* To this end, suppose that $\langle X; R\rangle$ is a presentation of the group G. We say that a group F is free on X if it can be presented in the form $F = \langle X; \ \rangle$ (notice that it has an empty set of relators). One can then think of G as the quotient of the free group F on X by the smallest normal subgroup K of F containing R: $G \cong F/K$. It turns out that every element in a free group can be expressed uniquely as a *reduced X-word*, i.e., an X-word in which no consecutive pair of letters are inverses. The number of letters involved is termed the length of the X-word. One says that $f\colon \mathbb{N} \to \mathbb{R}$ is a *Dehn function* for this presentation of G if it satisfies the following condition:

for any reduced word w in F which takes on the value 1 in G, there are words $r_i \in R$, $p_i \in F$, and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, N$ such that

$$w = \prod_{i=1}^{N} p_i r_i^{\varepsilon_i} p_i^{-1} \text{ in } F \text{ and } N \leq f(l(w)).$$

It is not hard to see that if the function f, above, can be chosen to be a polynomial of degree $d \geq 1$, then there exists a Dehn function which is a polynomial of the same degree d for every finite presentation of G. Thus the existence of such a Dehn function is independent of the choice of finite presentation of G. One then says that G satisfies a *linear*, *quadratic*, *cubic*, etc., *isoperimetric inequality* if it has a finite presentation with a linear, quadratic, cubic, etc., Dehn function. In his fundamental paper on word-hyperbolic groups, Gromov proved the remarkable fact that *a finitely presented group is word-hyperbolic if and only if it satisfies a linear isoperimetric inequality.* Word-hyperbolic groups include all finite groups, all finitely generated free groups, cocompact groups of isometries of n-dimensional hyperbolic space, various classes of small cancellation groups, as well as the free product of any pair of hyperbolic groups. Thus all of these groups are automatic. Somewhat surprisingly, the direct product of two infinite hyperbolic groups is never hyperbolic, although the direct product of two automatic groups is again automatic. So every finitely generated abelian group is automatic. On the other hand, *a finitely generated nilpotent group is automatic if and only if it contains a subgroup of finite index which is abelian.* The very definition of an automatic group suggests that its structure is reasonably uncomplicated. This is manifested by the fact that *automatic groups are finitely presented*, that they *have a solvable word problem*, and that they *satisfy a quadratic isoperimetric inequality.* In addition to these properties, automatic groups have a property akin to that used initially to define word-hyperbolic groups. Indeed, if G is an automatic group which is generated by the finite set X, then one thinks of an X-word $w = b_1 \cdots b_n$ as a map from the set of nonnegative integers $0, 1, \ldots$ into G by setting $w(t) = \overline{b_1 \cdots b_t}$ $(t \leq n)$, $w(t) = \bar{w}$ $(t > n)$, where $\bar{w}$ denotes the value in G of the word w. Two X-words are called *k-fellow travellers* if for all t, $d(u(t), v(t)) \leq k$ in Γ. Automatic groups can be characterized using this " k-fellow traveller" notion.

This then, and a great deal more, is what this book is about. It is the end result of a preprint written by D. B. A. Epstein in 1987. This preprint, as well as subsequent versions, was widely circulated and, like Gromov's paper, created a great deal of interest and excitement. Parts of it were written by Holt and Thurston. The book itself was written by Epstein. It is divided into two parts. The first is an introduction to automatic groups, and the second is devoted to a number of topics in the theory of automatic groups. Chapters 1 and 2 contain a careful introduction to finite-state automata and regular languages, the definition of an automatic group, and a discussion of ways of "improving" the automatic structure of an automatic group. Some of the geometric aspects of the Cayley graph are discussed in Chapter 3, followed by a discussion of some examples of automatic groups in Chapter 4. Chapter 5 is devoted to obtaining a collection of axioms, which refer to a finite number of finite-state automata, which are satisfied if and only if these automata are the automata of some automatic group. These axioms are all expressed in terms of regular predicates, and so they can be checked algorithmically. This leads, in Chapter 6, to the description of a method for finding an automatic structure of any given finitely presented group which is an automatic group. The method depends on an implementation of the Knuth-Bendix procedure and provides the basis for a collection of computer programs—due to Epstein, Holt, and Sarah Rees—which actually finds an automatic structure of any given finitely presented group, provided it has one (of a slightly restricted kind). These programs have been surprisingly effective in a number of specific instances. Chapter 7 treats a generalisation of automatic groups, *asynchronously automatic groups*; and the first part is concluded with Chapter 8, which deals with nilpotent groups. Part 2 begins with the proof that the Braid group is automatic, while Chapter 10 involves "higher-dimensional, isoperimetric inequalities" and the proof that most of the special, linear groups are not automatic. Chapter 11 deals with geometrically finite groups, and the book concludes with the proof that significantly many of the fundamental groups of three-dimensional manifolds are automatic.

As I said at the outset, this is a remarkable book, and I recommend it very strongly to everyone who is interested in either group theory or topology, as well as to computer scientists. The exposition is thorough, if a little uneven. Some of the material is quite technical, requiring more than a little knowledge of three-dimensional topology. A few of the arguments presented are harder to understand than they need to be, and the inexperienced reader might well be discouraged at times. Despite these very minor shortcomings, I would urge such readers—indeed, all readers—to persevere. This is a very important piece of work, which contains a lot of lovely mathematics and lots of important ideas. It seems very likely to have a great impact on the way that we approach parts of combinatorial group theory as well as on the way that we deal with the burgeoning field of computational group theory.

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Infinite dimensional Lie superalgebras, by Y. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, and M. V. Zaicev. de Gruyter Expositions in Mathematics, vol. 7, Walter de Gruyter, Berlin and New York, 1992, 250 pp., $89.00. ISBN 3-11-012974-4

During the last two decades the theory of Lie superalgebras became an established part of algebra; its maturity is demonstrated by important applications in physics as well as by numerous interesting questions in ring theory which arose from studying Lie superalgebras and their universal enveloping algebras. Much of the literature has been devoted to finite-dimensional $\mathbb{Z}_2$-graded superalgebras. V.G. Kac's classification of simple Lie superalgebras [2] and M. Scheunert's monograph [6] are among the most notable references. As in the case of Lie algebras, the study of superalgebra representations is one of the focal points; since representations of a superalgebra naturally correspond to modules over its associative enveloping algebra ring-theoretic properties of universal enveloping algebras take the center stage in the representation theory. In some cases there is also a connection between identities holding in the enveloping algebra and representations of the Lie algebra (a similar link has also been exploited in studying certain group algebras).

Following the success in using $\mathbb{Z}_2$-graded Lie superalgebras to describe symmetries in quantum field theory, a natural generalization of the classical Lie superalgebra structure was introduced and studied in several special cases (see [7, 4, 5]).

Let $L = \bigoplus_{g\in G} L_g$ be an algebra over a commutative ring K with 1, graded by an abelian group G, and let $\varepsilon : G \times G \to K^*$ be a bilinear and alternating