are not automatic. Chapter 11 deals with geometrically finite groups, and the book concludes with the proof that significantly many of the fundamental groups of three-dimensional manifolds are automatic.

As I said at the outset, this is a remarkable book, and I recommend it very strongly to everyone who is interested in either group theory or topology, as well as to computer scientists. The exposition is thorough, if a little uneven. Some of the material is quite technical, requiring more than a little knowledge of three-dimensional topology. A few of the arguments presented are harder to understand than they need to be, and the inexperienced reader might well be discouraged at times. Despite these very minor shortcomings, I would urge such readers—indeed, all readers—to persevere. This is a very important piece of work, which contains a lot of lovely mathematics and lots of important ideas. It seems very likely to have a great impact on the way that we approach parts of combinatorial group theory as well as on the way that we deal with the burgeoning field of computational group theory.

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Infinite dimensional Lie superalgebras, by Y. A. Bahturin, A. A. Mikhalev, V. M. Petrogradsky, and M. V. Zaicev. de Gruyter Expositions in Mathematics, vol. 7, Walter de Gruyter, Berlin and New York, 1992, 250 pp., $89.00. ISBN 3-11-012974-4

During the last two decades the theory of Lie superalgebras became an established part of algebra; its maturity is demonstrated by important applications in physics as well as by numerous interesting questions in ring theory which arose from studying Lie superalgebras and their universal enveloping algebras. Much of the literature has been devoted to finite-dimensional $\mathbb{Z}_2$-graded superalgebras. V.G. Kac's classification of simple Lie superalgebras [2] and M. Scheunert's monograph [6] are among the most notable references. As in the case of Lie algebras, the study of superalgebra representations is one of the focal points; since representations of a superalgebra naturally correspond to modules over its associative enveloping algebra ring-theoretic properties of universal enveloping algebras take the center stage in the representation theory. In some cases there is also a connection between identities holding in the enveloping algebra and representations of the Lie algebra (a similar link has also been exploited in studying certain group algebras).

Following the success in using $\mathbb{Z}_2$-graded Lie superalgebras to describe symmetries in quantum field theory, a natural generalization of the classical Lie superalgebra structure was introduced and studied in several special cases (see [7, 4, 5]).

Let $L = \bigoplus_{g\in G} L_g$ be an algebra over a commutative ring K with 1, graded by an abelian group G, and let $\varepsilon : G \times G \to K^*$ be a bilinear and alternating (i.e., $\varepsilon(g, h) = \varepsilon(h, g)^{-1}$) form. Such a form gives rise to modified versions of the "super-anticommutative" and "super-Jacobi" identities:

$$[x, y] = -\varepsilon(g, h)[y, x] \text{ and } [[x, y], z] = [x, [y, z]] - \varepsilon(g, h)[y, [x, z]]$$

for any $x \in L_g$, $y \in L_h$, $z \in L$. L is called a *color Lie superalgebra* if both of these identities hold in it. Color superalgebras obviously include the class of Lie algebras (when G is trivial) as well as $\mathbb{Z}_2$-graded Lie superalgebras. Examples in [4] show that the generalization is nontrivial and interesting to physicists.

The book under review is a thorough survey of color Lie superalgebras, their varieties, and their enveloping algebras. Many results are contained in the authors' earlier (sometimes unpublished) work, which makes this exposition a unique reference on the subject.

The first chapter contains a terse review of basic results on Lie algebras, graded and filtered algebras, and Lie superalgebras. We are also treated to several analogues of classical results such as Schur's Lemma and Burnside's theorem. Much attention is given to varieties of color superalgebras; for example, using earlier results due to Bahturin, the authors prove that identities of every variety of metabelian (i.e., satisfying $[[x, y], [z, t]] = 0$) color Lie superalgebras are finitely based. This is a partial analogue of a well-known consequence of Lie's classical theorem for finite-dimensional solvable Lie algebras in characteristic zero.

Chapter 2 is devoted to the study of free color superalgebras and their graded subalgebras. Graded subalgebras of free color superalgebras are shown to be free, and an analogue of Schreier's formula is proved as one of the consequences. Drawing on Shirshov's regular word technique, the authors actually construct bases of free objects in varieties of color superalgebras in both zero and positive characteristics.

Next, the authors discuss the structure of the enveloping algebra of a color superalgebra and of the *restricted universal enveloping algebra* of a color p-superalgebra (i.e., a generalization of the notion of a restricted Lie algebra to the G-graded case). G. Bergman's "Diamond Lemma", which contains the essential ideas in Birkhoff's proof of the Poincaré-Birkhoff-Witt (PBW) theorem for enveloping algebras of Lie algebras, is used to verify the expected versions of the PBW theorem for color enveloping algebras. This in turn serves to show that the enveloping algebra $\mathcal{U}(L)$ of a color Lie superalgebra is endowed with a Hopf algebra structure (this, as many other results, requires the additional assumption that the characteristic of the base ring is distinct from 2 and 3). In the case of characteristic zero, the primitive elements of $\mathcal{U}(L)$ coincide with L.

Special attention is given to the problem of algorithmic decidability of equations given defining relations, with Shirshov's composition lemma for regular words being the foundation of the crucial proofs. An entire chapter is then devoted to establishing criteria for the enveloping algebra (and the restricted enveloping algebra) of a Lie color superalgebra graded by a finite group to be a PI-ring. Corresponding results for the enveloping algebra of a Lie algebra were obtained by Latyshev and Bahturin.

The authors then proceed to investigate the Jacobson radical of color enveloping algebras. They characterize color superalgebras whose irreducible representations have bounded dimensions and obtain several results on the structure of the restricted enveloping algebra $u(L)$ of a p-superalgebra, which to some extent parallel well-known facts about group rings: for example, that $u(L)$ is self-injective if and only if L is finite dimensional.

The last chapter focuses on the relationship between identities of a color superalgebra over a field and certain finiteness conditions such as residual finiteness (being a subdirect product of finite-dimensional superalgebras), "matrix presentability" in the sense of A.I. Mal'cev, the Noetherian property, and the weaker *Hopfian* condition (which means that for every endomorphism of L, "onto" implies "one-to-one"). The study of links between these properties, dating back to Mal'cev [3], has yielded several well-known results on varieties of associative algebras over an infinite field (see [1]). The authors develop similar connections between these properties holding locally (i.e., for finitely generated algebras) in varieties of color Lie superalgebras.

The monograph does not make for easy reading. In several crucial places the notation seems a little sloppy. Some definitions are "understated" and leave the reader guessing. Even though each chapter is preceded by a few general remarks, the sense of direction is often missing. Some of the machinery employed by the authors is heavy and nonintuitive: a more extensive commentary would help make the reading smoother. Even though the bibliography is quite complete, references in the text are sometimes spotty. For instance, the generic flatness lemma is described as taken from Dixmier's book, and there is no citation for Quillen's well-known original paper.

The reviewer also felt that the book would significantly benefit from a more thorough prepress scrutiny. Occasional linguistic lapses make the text harder to read than necessary. Clumsy phrases such as ". . . we are doomed to have $\varepsilon(g, h)\varepsilon(h, g) = 1$" (p. 14) soon become distracting rather than amusing and could easily have been avoided given a more careful editorial review.

Despite these cosmetic flaws, the book will be a valuable reference for every mathematician working in this relatively young area. It can also be recommended to advanced graduate students, who may gain a lot from such intense exposure to a wide range of sometimes unorthodox techniques and ideas.

References

[1] A.Z. Anan'in, *Locally finitely approximable and locally representable varieties of algebras*, Algebra and Logic **16** (1978), 1–16.

[2] V.G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[3] A.I. Mal'cev, *On representations of infinite dimensional algebras*, Mat. Sb. **13** (1943), 263–285.

[4] V. Rittenberg and D. Wyler, *Sequences of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded Lie algebras and superalgebras*, J. Math. Phys. **19** (1978), 2193–2200.

[5] ———, *Generalized superalgebras*, Nuclear Phys. B **139** (1978), 189–202.

[6] M. Scheunert, *The theory of Lie superalgebras. An introduction*, Lecture Notes in Math., vol. 716, Springer-Verlag, New York, Heidelberg, and Berlin, 1979.

[7] M. A. Vasil'ev, *deSitter supergravity with positive cosmological constant and generalized Lie superalgebras*, Classical Quantum Gravity **2** (1985), 645–659.

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