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BOOK REVIEW

The geometry of algebraic Fermi curves, by D. Gieseker, H. Knörrer, and E. Trubowitz. Academic Press, New York, 1992, vii+236 pp. ISBN 0-12-282620-5

In solid state physics there exists a simple quantum mechanical model of crystalline matter, which is based on an approximation (the independent electron approximation) described below. In this model the physical properties of a crystal are to a large extent determined by its so-called Fermi surface. For instance, physicists can predict from the shape of this surface whether the material will act like a conductor, semiconductor, or an insulator.

I shall now first explain the notion of Fermi surface for arbitrary dimension. Then I shall discuss its analog in the discrete setting of the book under review, which deals with dimension two (the Fermi surface, a hypersurface in $\mathbb{R}^3$ gets replaced by the Fermi curve, a hypersurface in $\mathbb{R}^2$). A detailed account of the physical aspects can be found in [AM].

Suppose that one places rigid ions at the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Suppose in addition that there are electrons which move independently from each other under the influence of a potential $q(x) \in L^2(\mathbb{R}^d)$, which is periodic with respect to this lattice and which describes the potential energy of the entire system consisting of ions and electrons. Quantum mechanics then tells you that the probability of finding an electron at x is described by a complex wave function, which can be written as a superposition of those eigenfunctions $\Psi(x)$ of the Schrödinger operator $-\Delta + q(x)$ that are quasi-periodic; i.e., for some $k \in (\mathbb{R}^d)^*$ one has

$$\Psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\Psi(x), \quad \forall \gamma \in \mathbb{Z}^d .$$

In mathematical terms, fixing k, the *crystal momentum*, describes a selfadjoint boundary value problem with discrete spectrum $E_1(k) \le E_2(k) \le \cdots$. The function $E_j(k)$ is continuous and periodic in the lattice dual to $\mathbb{Z}^d$. The unit hypercube Q serves as a fundamental domain for this lattice.

Suppose now that there are $N_L = L^d n$ electrons in $L \cdot Q \subset \mathbb{R}^d$ where n, the electron density, is independent of $L \in \mathbb{Z}_{>0}$ (the crystal size). By the Pauli exclusion principle, valid for electrons (they are fermions), exactly two electrons can be placed in one energy level (with opposite spin), so the lowest energy level for the system of N_L electrons in the box is realized when the energies of these electrons are the first $[N_L/2]$ eigenvalues with highest value E_L, say. Now, if we restrict attention to wave functions that are periodic with respect to the lattice $L \cdot \mathbb{Z}^d$, the values for k are restricted to $\frac{2\pi}{L} \cdot \mathbb{Z}^d \subset (\mathbb{R}^d)^*$; and its seems quite likely (and in fact is true) that E_L tends to some finite limit if L grows to infinity. This limit E_n

only depends on the electron density and is called the *Fermi-energy*. The locus in k-space corresponding to states with the Fermi-energy forms a hypersurface in $(\mathbb{R}^d)^*$, the *Fermi-hypersurface*. In the physical three-dimensional world this gives the Fermi-surface, and its physical meaning is that it separates occupied states from nonoccupied states at absolute temperature zero.

In this model another experimentally observable quantity plays an important role, the (*integrated*) *density of states function* $\rho(t)$. To compute it, one counts the number of electrons of energy less than t per unit volume in the box $L \cdot Q$ and takes the limit for L going to infinity.

One of the problems now is whether knowledge of "sufficiently many" Fermi-hypersurfaces is enough to reconstruct the potential. Indeed, as noted before, in the real world one can make several predictions about the crystal from measuring the Fermi-surface alone. An even more challenging question is whether the potential can be reconstructed from measuring the density of states function for "enough" energy-levels.

Due to periodicity, the Fermi-hypersurface can be viewed as a hypersurface of the d-torus $(S^1)^d$; moreover, we can also define the hypersurfaces $F_t \subset (S^1)^d$ corresponding to any energy-level $t \in \mathbb{R}$ and collect them into a hypersurface $B(q)$ in $(S^1)^d \times \mathbb{R}^1$.

In general these varieties are hard to study, so one introduces a suitable discrete model. In order to be able to apply better the methods of algebraic geometry, one also works over the complex numbers, substituting $(\mathbb{C}^*)^d \times \mathbb{C}$ for $(S^1)^d \times \mathbb{R}^1$ and using complex-valued potentials q. Furthermore, one replaces $\mathbb{R}^d$ by $\mathbb{Z}^d$ and the ion lattice $\mathbb{Z}^d$ by

$$\Gamma = \bigoplus_{j=1}^{d} \mathbb{Z} \cdot a_j \vec{e}_j$$

where $\vec{e}_j$ is the jth unit vector, and one assumes that the a_j are distinct prime numbers. One chooses a fundamental domain F for Γ. The vector space of complex-valued Γ-periodic functions on $\mathbb{Z}^d$ equipped with the inner product

$$\langle \varphi, \psi \rangle = \frac{1}{a_1 \cdots a_d} \sum_{x \in F} \varphi(x) \overline{\psi}(x)$$

is denoted by $L^2(\mathbb{Z}^d/\Gamma)$. Potentials $q(x)$ are supposed to belong to this space. To formulate the analogous spectral problem, one introduces the shift-operators S_j which act on functions in $L^2(\mathbb{Z}^d/\Gamma)$ by

$$S_j f(x_1, \ldots, x_j, \ldots, x_d) = f(x_1, \ldots, x_{j+1}, \ldots, x_d).$$

Now, the discrete Laplacian is given by $\Delta = \sum_{j=1}^d S_j + \sum_{j=1}^d S_j^{-1}$. This discrete problem then translates into

$$(*) \qquad \begin{cases} (-\Delta + q - t)\psi = 0 & (t \in \mathbb{C}) \\ S_j^{a_j} \psi = \xi_j \psi & (\xi_j \in \mathbb{C}^*),\ j = 1, \ldots, d, \end{cases}$$

and one introduces

$$B(q) = \{(\xi_1, \ldots, \xi_d, t) \in (\mathbb{C}^*)^d \times \mathbb{C} \mid \exists \psi \neq 0,\ \psi \text{ solves } (*)\}.$$

The fibres F_t over $t \in \mathbb{C}$ are the complex analogs of the Fermi-hypersurface as introduced before. In fact, for t equal to the Fermi-energy, the real hypersurface

$$\phi_t \underset{\text{def}}{=} S^d \cap F_t$$

with $S^d \in (\mathbb{C}^*)^d \times \{t\}$ is the precise analog in the discrete setup.

It is not hard to see that $B(q)$ is the zero-set of a polynomial P of degree $a_1 \cdots a_d$ in the variables ξ_j, ξ_j^{-1}, $j = 1, \dots, d$ and t, and so is algebraic. One can also show that the nonintegrated density of states function (the derivative of$\rho(t)$) is the period integral

$$\int_{\phi_t} (-1)^j \frac{\frac{\partial}{\partial t} P \, d\xi_1 \cdots \wedge \widehat{d\xi_j} \wedge \cdots d\xi_d}{\xi_1 \cdots \xi_d \frac{\partial}{\partial \xi_j} P} .$$

Recall that $\phi_t = F_t \cap S^d$ is the real Fermi-hypersurface and that for t fixed $P(\xi, t) = 0$ defines F_t. Also note that the $(d-1)$-form over which one integrates can be identified as the restriction of F_t of the global $(d-1)$-form ω on $(\mathbb{C}^*)^d \times \mathbb{C}$ defined by

$$\pi^*(dt) \wedge \omega = \frac{d\xi_1}{\xi_1} \wedge \cdots \wedge \frac{d\xi_d}{\xi_d},$$

where π denotes the projection onto the complex t-line. So also the density of states function can be identified canonically as a period integral over a naturally defined regular $(d-1)$-form on F_t, and one is in the position to apply algebraic geometry.

In the book under review the two questions formulated previously are solved for the discrete model and for “generic” real-valued potentials if $d = 2$, i.e., for Fermi-curves. Amazingly, for such potentials it is true that the density of states function-germ near a real point (but viewed as a holomorphic function) determines q up to some obvious ambiguities (changes of signs in the coordinates and translations in the coordinates). This statement, in fact, is a conjunction of two theorems: one, saying that for generic potentials q the total space $B(q)$ determines the potential up to the aforementioned ambiguities, the other stating that for real-valued generic potentials the germ of the density of states function near a real point determines $B(q)$.

The first theorem (Theorem 4.4 on page 55) is much easier to prove than the second one (Theorem 13.1 on page 222) but already involves some algebraic geometry. The variety $B(q)$ is closed up inside $\mathbb{P}^2 \times \mathbb{P}^1$, and its singularities as well as the singularities of all the compactified curves are carefully analysed; after this analysis Theorem 4.4 follows quite painlessly. At this point I would advise the reader to read first the argument used to prove Lemma 3.3, since the same sort of reasoning is implicit in the proof of Theorem 4.4 and clarifies the proof considerably.

The second theorem, however, involves some deep algebraic geometry (Torelli's theorem for curves; see, e.g., [G-H] and Deligne's theorem on the fixed part [De, 4.1.2]) and also a lot of detailed topological considerations related to deformation theory of isolated singularities. In fact, in order to study the generic potential, the authors first make a detailed study of the potential zero case and the case of separable potential, $q(x) = q_1(x_1) + q_2(x_2)$. Then they investigate how the situation changes when you deform away from these special cases; in particular, they make a careful study of the resulting vanishing cycles and the monodromy action. This is the hardest part of the book and comprises over two-thirds of it!

Among the potential readers one should certainly count the algebraic geometer as well as the mathematical physicist working in solid state physics.

Indeed, the *methods and theory* borrow heavily from algebraic geometry; and to motivate such a reader, the first chapter of the book is indispensable. I should, however, give a warning. The usual notions of "Fermi-energy" and "Fermi-surface" as adopted by physicists are those I gave above, and these differ from those given on page 5 in the book under review.

The *topic*, however, should attract the second kind of potential reader. Such a reader, of course, is well acquainted with the notions of "Fermi-surface", "density of states function", and the like. He or she probably knows very little algebraic geometry. In order to bridge this gap the authors have included a detailed description of what happens in the one-dimensional case where everything is much simpler and explicitly calculable. In particular, one can view this case as a toy model for the proof of Theorem 4.4. The calculations in the subsequent sections have been carried out in great detail and have been illustrated with many pictures and diagrams.

From Chapter 8 on, however, the authors are more demanding. For instance, in Chapter 8 the notion of "vanishing cycle" is not fully explained; a small computation showing that the intersection of a small ball about an ordinary double point with a nearby fibre has the homotopy type of a circle would have been helpful. Also, a reference could have been provided for the basic Picard-Lefschetz formula, and some explanation could have been offered for the extra factor of 2 in this formula when the total space acquires a node (page 166). Later on (page 192) the reader is supposed to be familiar with the theory of semiuniversal deformations, at least for double points of type A_2 (hopefully the reader understands that those are the cusp singularities treated in the preceding lines). Finally, the two difficult theorems mentioned before are just referred to; they are not stated, nor is it explained how these can be used to draw the crucial conclusions on page 224 (proof of Proposition 13.1 and Proposition 13.2). This makes reading hard for novices in this matter and could have been easily avoided. See for instance [P], where this last part is explained in more detail.

From the preceding description of the contents it should be clear that the book is a difficult and deep piece of mathematics entirely devoted to the proof of two basic theorems. It is basically an expanded research article. Apart from the few criticisms given, I found the book very readable. Anyone who wants to see an intricate piece of highly nontrivial algebraic geometry applied to a crude model of the "real world" should find it rewarding.

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