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Typical singularities of differential 1*-forms and Pfaffian equations*, by Michail Zhitomirskii. American Mathematical Society, Providence, RI, 1992, xi+176 pp., $116.00. ISBN 0-8218-4567-5

A contact structure on a $(2n+1)$-dimensional manifold M is a field of tangent hyperplanes which is generic in the following sense: locally near each point this field may be defined by a 1-form ω such that $\omega \wedge (d\omega)^n \neq 0$ everywhere. Contact geometry provided by a contact structure is the "odd-dimensional" frame for mechanics. It occurs naturally when one adds the variable "times" to positions and impulses and is the natural basis for optics and wave propagation. For instance, let B^n be a Riemannian manifold, T^*B^n its cotangent bundle, and PT^*B^n its projectivized cotangent bundle. A point in PT^*B^n is a tangent hyperplane, also called contact element. This space carries a natural contact structure defined as follows: a tangent vector v at $\alpha \in PT^*B^n$ belongs to the contact hyperplane τ_α if and only if the projection of v on the tangent space of B^n belongs to α. Now if $S^{n-1} \subset B^n$ is a hypersurface of B^n, wave fronts defined by equidistant hypersurfaces are projections of n-dimensional integral submanifolds of contact structure of PT^*B^n (such an integrable submanifold of maximal dimension n is called a *Legendre submanifold.* Typical examples of Legendre submanifolds are the fibers of the projection: $PT^*B^n \to B^n$). See, for instance, [A1, AKN] for a geometric approach of contact geometry and its applications.

Zhitomirskii's monograph deals with a generalized contact geometry in the sense that he replaces the contact structure by a Pfaffian equation, locally given by $\{\omega = 0\}$ where ω is any 1-form, not necessarily regular as above: the manifold M may have an even dimension, and the form $\omega \wedge (d\omega)^n$ may be zero at some points of M.

A motivation to look at general Pfaffian equations can be found in a result of Arnold and Givental [AG]: any germ of a submanifold of a contact manifold is locally defined (up to contactomorphism) by the restriction of the contact structure to the submanifold. This means that the classification of germs of submanifolds in a contact manifold, up to contactomorphisms, reduces to the classification of Pfaffian equation germs up to diffeomorphism equivalence of 1-forms, i.e., up to local diffeomorphism of the submanifold and multiplication by nonvanishing function germs. The classification of Pfaffian equation germs is precisely the principal aim of the monograph.

Zhitomirskii limits himself to typical Pfaffian equation germs on manifolds M, i.e., to germs which cannot be eliminated by small perturbations or, more precisely, which are of codimension less than the dimension of M. He considers also the classification of 1-form germs, up to diffeomorphism. Later we return to applications of this Pfaffian equation and 1-form classification results, as they are briefly presented in appendices of the monograph, but first we concentrate on the classification results.

The simpler germs are the finitely determined ones because they are determined by a finite jet and so depend just on a finite number of parameters.

A first result of Zhitomirskii is the computation of the codimension of germs which are not finitely determined: for Pfaffian equations, such germs do not exist if $\dim M = 3$, and they exist generically in codimension 3 if $\dim M = 4$ and in codimension 4 if $\dim M \geq 5$. The situation is, of course, worst for 1-forms where the nonfinitely determined germs may arrive in codimension 2. Nonfinitely determined germs cannot be characterized by any finite jet. They depend on functional moduli described in the text.

The monograph is essentially devoted to the classification of finitely determined germs of Pfaffian equations and 1-forms. The author succeeds in presenting the complete catalogue of such germs, each of them given by a simple polynomial model depending on a minimal number of parameters, which are the classification invariants. This catalogue is rather impressive, because it contains seventeen different types of Pfaffian equations and six types of 1-forms. Happily, a synthetic presentation of these results is made in the first chapter.

Next, each particular case (Pfaffian equation in odd dimension, Pfaffian equation in even dimension, and so on) is studied in a particular chapter with all proofs and detailed results. For instance, results about adjacencies of typical singularities are found in the specialized chapters.

First classification results were obtained by G. Darboux. Generically for a Pfaffian equation ω on M, the equation $\omega \wedge (d\omega)^{n-1} = 0$ defines a subset M_1 of codimension 1 if $\dim M = 2n - 1$ and of codimension 3 if $\dim M = 2n$, for $n \geq 2$. In the first case ω defines a contact structure on $M - M_1$. The Darboux theorem says that this contact structure is locally C^∞ equivalent to the canonical one:

$$dp_1 + \sum p_1 dq_1 = 0 \tag{1}$$

in coordinates $p_1, p_2, q_2, \ldots, p_n, q_n$ of $\mathbb{R}^{2n-1}$. So all the contact structures are locally equivalent, and the contact geometry is locally "flat". A similar result holds if $\dim M = 2n$: the equation $\omega = 0$ is generically equivalent to (1) in coordinates $p_1, q_1, \ldots, p_n, q_n$ of $\mathbb{R}^{2n}$.

All other possible stable germs are classified in the monograph. Some of them depend on a parameter. This is not in conflict with their stability, because the corresponding singularities are not isolated. In each case a simple model of degree less than 3 is given.

As stated before, the author presents a complete list of all finitely determined Pfaffian equations and 1-form germs. Some of these results were previously obtained by Lychagin [L] and the author himself, but others appear for the first time here. In each case an explicit polynomial model is given. When the dimension $n = 2k$ or $2k+1$ with $k \geq 2$, this model has the following structure: a polynomial form of degree less than 5, independent of any parameter, plus an exact form dG_μ where G_μ is a quadratic Hamiltonian function and $\mu = (\mu_1, \ldots, \mu_k)$, the parameter is essentially the collection of eigenvalues of the corresponding Hamiltonian vector field.

To give a clear idea of these results, let us consider in more detail the generic Pfaffian equations in dimension 3. Because one is just interested in local classification, one can suppose that the Pfaffian equation is given by a 1-form ω. One also assumes that ω is generic, as will be made precise later. The general result is that the germ of $\{\omega = 0\}$ is finitely determined at any point. A detailed description is as follows: the equation $\{\omega = 0\}$ defines a smooth surface M_1.

At any point of $M - M_1$ the form of the Pfaffian equation is equivalent to the Darboux model $dz + ydx = 0$. Next, the form ω is zero at some isolated points of M_1; these points form a subset M_0. Another subset M_2 of M_1 is made by the points where $\omega \neq 0$ and $\operatorname{Ker}\omega = TM_1$. On $M_1 - M_0 \cup M_2$, the plane $\operatorname{Ker}\omega$ is transversal to TM_1, and the intersection $\operatorname{Ker}\omega \cap TM_1$ defines a field of lines 1 on $M_1 - M_0 \cup M_2$. Germs at these points were studied by J. Martinet, who proved they are equivalent to $dy + x^2dz$ (the local equation for M_1 is $x = 0$); the germ at a point of M_0 is equivalent to

$$zdz + 1/2(xdy - ydx) + dG_\mu(x, y) = 0 \tag{2}$$

where G_μ is the quadratic Hamiltonian $\frac{1}{2}\mu xy$ or $\frac{1}{2}\mu(x^2 + y^2)$ (note that the germ of ω is no longer stable but is 1-determined). This result is new. At each point of M_2, the germ is equivalent to one of the following:

$$dy + (xy + x^2z + bx^3z^2)dz = 0 \tag{h}$$

or

$$dy + (xy + x^3/3 + xz^2 + bx^3z^2)dz = 0. \tag{e}$$

These germs are 5-determined and depend on the parameter b.

To prove that the parameter μ in (2) or b in (h), (e) is invariant for equivalence, one has to look more closely at the local geometry. First, observe that near each singular point of $\omega|M_1$ a well-defined singular foliation, i.e., a local vector field X_ω up to a multiplicative function, extends the line field l (we just have to take any area form Ω on M_1 and define X_ω by $X_\omega \rfloor \Omega = \omega$). At each singular point of X_ω (i.e., on $M_0 \cup M_2$) the ratio of eigenvalues is an invariant independent of the different choices. For instance, it is equal to $(1-\mu)/(1+\mu)$ for (2): this proves that the parameter μ is an invariant (the unique one). At the points of M_2, X_ω has a resonant linear part equivalent to $x\partial/\partial x - z\partial/\partial z$ in the case (h) and to $x\partial/\partial z - z\partial/\partial x$ in the case (e). So no invariant is related to it. On the other hand, the coefficient b is related to a holonomy invariant of the foliation: for instance, in the case (e) the singular point is a focus of X_ω with return map $u \to u + u^2 + \alpha u^3$, and α is related to b.

The method used to obtain the classification results is the homotopy method introduced by Moser in [M] and intensively used for applications by Mather [Mat] and for differential forms and Pfaffian equations by Martinet [Mar]. A general reference given in the monograph is [AVG]. The idea for this method is as follows: if one wants to compare two germs of differential forms ω, ω_1 at some point, one writes the 1-parameter family $\omega_t = \omega + t(\omega_1 - \omega)$ linking $\omega = \omega_0$ to ω_1 and looks for a 1-parameter family of diffeomorphism germs h_t verifying $h_{t*}\omega = \omega_t$. By derivation, this equation is equivalent to finding a 1-parameter family of vector fields v_t solution of the linear equation:

$$v_t \rfloor d\omega_t + d(v_t \rfloor \omega_t) = \omega - \omega_1. \tag{H}$$

The possibility of solving this last equation depends strongly on ω, ω_1. First, one needs in general to have a second member $\omega - \omega_1$ as flat as possible. This will be obtained by fixing a k-jet of ω: if one wants to prove that ω is k-determined, one can choose ω_1 such that $j^k(\omega - \omega_1) = 0$. This assumption will also imply that the t-dependence in the first member is of order bigger than $k + 1$; so if some generic assumption is made on the k-jet of ω, it remains valid on ω_t for any value of t.

In the regular case (Darboux and Martinet theorems) the equation (H) admits a unique geometrical solution which is also analytic. It is not the case for the singular cases where the method consists first of solving the problem formally. Then the question is reduced to an equation on flat functions for which more specific methods were already developed, for instance by Belitskii [Be]. In the monograph the author introduces a new, useful improvement, a relative version of this homotopy method, working relative to some projection P, so that the solution is found up to some term in $\operatorname{Ker} P$.

Proofs of some new results are very intricate. For instance, the above-mentioned result for points in M_2 for Pfaffian equations in dimension 3 needs nine pages of proof involving a succession of normalizations for finite jets before one is able to attack the final step on flat functions.

The text is rather technical. Happily the author has provided some interesting applications of his classification results. I have already mentioned the connection obtained by Givental between Pfaffian equations and contactomorphisms. The local classification of first-order partial differential equations is another fine field of applications: a germ of a first-order partial differential equation $F(x_1, \dots, x_n, u, \partial u/\partial x_1, \dots, \partial u/\partial x_n) = 0$ is treated as a germ of a hypersurface $E\colon F(x_1, \dots, x_n, u, y_1, \dots, y_n) = 0$ in the contact space $(\mathbb{R}^{2n+1}, \omega)$, where $\omega = du - \sum y_i dx_i$. One can apply again Givental's result. Another field of application is control theory. One considers in $\mathbb{R}^n$ a controlled system $F(x) + \sum u_i G_i(x)$ where the G_i are the control vector fields and the u_i are the control functions, and one introduces a natural equivalence for control theory: the feed-back one. Then it appears that the feed-back equivalence between two germs of control systems reduces to the classification problem for the 1-form defined by $\omega(F) \equiv 1 \quad \omega(G_i) \equiv 0$ for $i = 1, \dots, n$ at least when $\dim(F, G_1, \dots, G_n) = n$. It would be interesting to see what are the consequences of existence of singularities in the control systems for typical questions, such as the accessibility problem, for instance.

This monograph is the first complete reference for the classification problem, and it contains a complete and well-documented list of all finitely determined germs of 1-forms and Pfaffian equations. For this reason it will surely be an indispensable reference for anybody who wants to understand this difficult theory. It will also be a valuable help for people just interested in applications, because the author has summarized his results in very clear and easily readable tables of explicit polynomial models for each of the finitely determined singularities. This work fills a gap in the present literature, and it will surely be very welcome for this reason.

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Designs and their codes, by E. F. Assmus, Jr., and J. D. Key. Cambridge University Press, London and New York, 1992, x + 352 pp., $69.95. ISBN 0-521-41361-3

The announcement several years back of the nonexistence of a projective plane of order ten brought both elation and a lingering sense of disappointment to the mathematics community. The disappointment was due in part to the "proof" being supplied by a computer. But perhaps more significant was the prevailing opinion that much of what was accomplished in solving this problem applied only to the plane of order ten. In effect, outside of some programming advances, we failed to "learn anything". However, one can argue very much to the contrary that the projective plane of order ten served as a catalyst to the community that coding theory can play a very significant role in the study of combinatorial designs. Indeed, much of what is contained in *Designs and their codes* speaks to the tremendous influence the plane of order ten has subsequently had on the analysis and classification of designs in a much broader context than projective planes. As the recent book of van Lint and Wilson [7] testifies, coding theory is finding a place in the mainstream of combinatorics. This new book by Assmus and Key is a welcome addition to a very exciting and relatively new application of an established discipline to combinatorics.

To bring the story into focus, we briefly recount the influence of coding theory on the resolution of the plane of order ten. Put simply, a projective plane of order ten is a collection of 111 subsets of size eleven from the set $\{1, 2, \ldots, 111\}$ with the special property that any two subsets have precisely one element in common. Generally, a projective plane of order n is a collection of n^2+n+1 subsets (called *lines*) of size $n+1$ from a set of *points* $\{v_1, v_2, \ldots, v_{n^2+n+1}\}$ with the special property that any two subsets have precisely one element in common.

Construction of a projective plane of order n is immediate (but often not unique) when n is a power of a prime: take as the (n^2+n+1)-set of points