cultural heritage of Middle Europe" invites some further counting. Indeed, about 75 percent of the recent entries are by authors with some Central European connection. In addition to the authors, we see frequent mention of Janyška, Krupka, Mauhart, Mikulski, Zajtz, and others.

Despite the burst of Central European activity, I hope that the whole mathematical community will take note of this book. It has no geographic limitations or preferences. In fact, a number of the seminal ideas, predating 1980, came from other parts of the world. Without these, the book might not have been written.

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The Schwarz function and its generalization to higher dimensions, by Harold S. Shapiro. University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 9, Wiley-Interscience, New York, 1992, xi+108 pp., $59.95. ISBN 0-471-57127-X

Looking at the contents of the book, one is tempted to suggest the alternative title: *Lectures on the Schwarz function and its potential in applications.* What is generalized to higher dimensions is not the Schwarz function itself but a related "Schwarz potential". Both the function and the potential play a role in a variety of topics. Prominent among the latter are so-called quadrature domains. These have been a special interest of the author and his coauthor, Dov Aharonov, and of, among others, Sakai [5, 6]. Perhaps of wider interest, at the hands of the author, the Schwarz function turns out to be a useful device in the study of various operators in complex analysis.

But what are the Schwarz function and the Schwarz potential?

Most mathematicians are familiar with H. A. Schwarz's reflection principle, which provides analytic continuation through reflection in a segment of the real axis. Specifically, let Ω be a domain in the upper half-plane $\operatorname{Im} z > 0$, part of whose boundary is a real open interval I. Let $f(z)$ be any holomorphic function on Ω which extends continuously to $\Omega \cup I$ and whose extension $f(x)$ to I is real. Then f can be continued analytically to the domain $\Omega \cup I \cup \overline{\Omega}$; the holomorphic continuation f^* is given by the functional equation

$$f^*(z) = \overline{f(\overline{z})}.$$

Here bars denote reflection in the real axis or complex conjugation.

It is perhaps less well known that Schwarz proved a similar result for analytic continuation across an arbitrary analytic boundary arc Γ. Just as in the case of the reflection $z \to \overline{z}$, the reflection function $R(z) = R_\Gamma(z)$ for an analytic arc Γ is antiholomorphic. Long ago a few mathematicians, D.-A. Grave (1895) in France and G. Herglotz (1914) in Germany, started to use the complex conjugate of Schwarz's reflection function which is holomorphic. However, it took Davis to coin the name Schwarz function $S(z)$ for $\overline{R(z)}$; cf. [1]. We proceed to a constructive definition of this Schwarz function.

Definition. Let Γ be a (nonsingular) real-analytic Jordan arc in $\mathbf{R}^2 \cong \mathbf{C}$ not including end points. Around any one of its points a, the arc may be represented in the form $F(z, \overline{z}) = 0$, where $F(z, w)$ is a holomorphic function of two variables with $\partial F/\partial w \neq 0$. Thus by the implicit function theorem, there is a unique (holomorphic) function $w = S(z)$ on a neighborhood of a such that $F(z, S(z)) = 0$, hence $S(z) = \overline{z}$ for z on Γ. Moving along Γ, one obtains a unique holomorphic function $S(z)$ on a neighborhood of the arc which takes the value $\overline{z}$ for z on Γ.

Beginning with this definition, the author shows that for z near Γ, the map $z \to R(z) = \overline{S(z)}$ defines a true reflection in Γ. That is, R is an involution which takes points from one side of Γ to the other side and leaves Γ itself invariant. Schwarz's reflection principle may now be stated in the following general form; cf. Nehari [4].

Theorem. *Let D be a plane domain part of whose boundary is an analytic arc σ, and suppose that f, continuous on $D \cup \sigma$, maps D conformally on a domain E for which $\tau = f(\sigma)$ is also an analytic boundary arc. Then f extends holomorphically across σ, and for points z close to σ but outside D, the extended function $\tilde{f}$ is given by*

$$\tilde{f}(z) = R_\tau(f[R_\sigma(z)]).$$

Analyzing the conditions which S or R have to satisfy, one finds that S may be represented in the form

$$S(z) = 2\partial w/\partial z = \partial w/\partial x - i\partial w/\partial y,$$

where w is a (real) solution of the Cauchy problem

$$\Delta w = 0 \text{ near } \Gamma, \qquad w - (x^2 + y^2)/2 \text{ and its gradient vanish on } \Gamma.$$

By the Cauchy-Kovalevskaya theorem, this problem has a unique (real-analytic) solution w on a neighborhood of Γ which Shapiro calls the *Schwarz potential* of the analytic arc. A related Cauchy problem is used later in the book to associate a generalized Schwarz potential with quadrature domains in $\mathbf{R}^n$, $n \geq 2$ (where the boundary might not be analytic).

Quadrature domains

A major part of the book is devoted to quadrature domains and quadrature identities. The latter have nothing to do with the quadrature formulas of numerical analysis such as Simpson's rule or the m-point Gauss formula on $[-1, 1]$ which is exact for all polynomials of degree $\leq 2m - 1$. The present formulas apply only to holomorphic or harmonic functions on special domains. Prototype is the mean value theorem for (integrable) harmonic functions u on a ball $B(0, R)$ by which the integral of u over the ball is equal to a constant multiple of $u(0)$. Another historical example concerns an oval of C. Neumann whose boundary is obtained by inversion of an ellipse. As a pendant to unusual potential theoretic "inwards balayage", the integral of harmonic u over the oval can be expressed in terms of the values of u at two "foci".

There are different notions of quadrature domains, but we will use those of the book. The test functions below are the integrable harmonic or holomorphic functions on the domain Ω (the latter only if $n = 2$).

Definition. A domain Ω in $\mathbf{R}^n$ is called a *quadrature domain in the wide sense* if there exists a distribution μ with compact support in Ω such that the quadrature identity

$$\int_\Omega u\,dx = \langle \mu , u \rangle$$

holds for all test functions u. Such a domain Ω is called a (proper) *quadrature domain* if $\operatorname{supp}\mu$ is a finite set, so that $\langle \mu , u \rangle$ reduces to a linear combination of values of u and derivatives of u at certain base points.

In $\mathbf{C}$ a large class of quadrature domains relative to holomorphic functions may be obtained with the aid of the residue theorem. These are the domains $\Omega = \phi(D)$ where D is the unit disc and ϕ is a rational function which is pole-free and injective on the closure $\overline{D}$; cf. Davis [1]. The author next shows that quadrature domains in the wide sense have an associated generalized Schwarz potential and, in $\mathbf{R}^2$, a generalized Schwarz function. He subsequently takes another look at Schwarz reflection. Topics are: the well-known reflection principle for harmonic functions in $\mathbf{C}$ which vanish on a boundary segment or on a more general analytic boundary arc, Study's geometric interpretation of such reflection in $\mathbf{C}^2$, and the failure of Schwarz reflection in $\mathbf{R}^3$.

Operators of complex analysis

Largely new material in the book shows that the Schwarz function can be used to advantage in certain problems involving operators of complex analysis. One chapter discusses the "Hilbert" and "Szegö" projectors for a bounded domain Ω in $\mathbf{C}$ with smooth boundary Γ. For Hölder continuous F on Γ, the Cauchy-type integral

$$\frac{1}{2\pi i}\int_\Gamma F(\zeta)(\zeta - z)^{-1}\,d\zeta$$

defines a holomorphic function $f_i(z)$ on the interior Ω of Γ, with boundary values $F_i(z)$, as well as a holomorphic function $f_e(z)$ on the exterior domain(s) with boundary values $F_e(z)$. The Plemelj-Sokhotski relations express the boundary values $F_i(z)$ and $F_e(z)$ in terms of $F(z)$ and the principal value of the Cauchy integral for z on Γ. By the theory of singular integrals, this principal value integral has the same qualitative behavior as the Hilbert transform. In particular, it maps $L^2(\Gamma)$ continuously into itself, and it preserves Hölder continuity of fractional order. It follows that the maps $F \to F_i$ and $F \to F_e$ for smooth F extend continuously to $L^2(\Gamma)$. The L^2 closure of the class of smooth functions F_i can be identified with the Hardy space $H^2(\Gamma) \cong H^2(\Omega)$. The counterpart for the functions F_e is $H_e^2(\Gamma)$, and one has the direct sum decomposition

$$L^2(\Gamma) = H^2(\Gamma) \oplus H_e^2(\Gamma).$$

The corresponding (oblique) projector $L^2(\Gamma) \to H^2(\Gamma)$ is here called the Hilbert projector H, while the orthogonal or Szegö projector $L^2(\Gamma) \to H^2(\Gamma)$ is called S. Kerzman and Stein have expressed S in terms of H:

$$S = H[I + (H - H^*)]^{-1},$$

and they proved that $S = H$ only for a disc [3]. The author obtains a number of stronger results of the latter type, e.g., $H - H^*$ has finite rank only for a disc. Besides the Schwarz function his approach involves the solution of the Dirichlet problem for Ω and F in terms of a double-layer potential

$$\frac{1}{2\pi} \int_\Gamma G(\zeta)(\partial/\partial n_\zeta) \log|\zeta - z|\, ds_\zeta .$$

The latter has boundary values from Ω which may be expressed as $\frac{1}{2}(H + CHC)G$ where C stands for complex conjugation. Setting $H + CHC = I + K$, one finally has to solve the equation

$$(I + K)G = 2F .$$

Fredholm's solution of the Dirichlet problem then uses the compactness of the operator K. In this context the author shows that Ω is a disc if K has finite rank.

Another chapter of the book is devoted to a certain operator on the Bergman space $L_a^2(\Omega)$ of the holomorphic functions in $L^2(\Omega)$, where at first the domain Ω in $\mathbf{C}$ is arbitrary. The operator will be related to boundary value problems for the biharmonic equation of elasticity which Friedrichs has treated with the aid of quadratic forms [2]. Here Shapiro introduces a "Friedrichs operator", defined as $T = PC$, where $P = P_\Omega$ is the orthogonal projector $L^2(\Omega) \to L_a^2(\Omega)$ and C stands for complex conjugation on the Bergman space. The spectral properties of the positive operator T^2 reflect the geometry of the subspaces $L_a^2(\Omega)$ and $CL_a^2(\Omega)$ within $L^2(\Omega)$. It may be derived from Friedrichs's results that T is compact if Ω is bounded and $\partial\Omega$ is smooth everywhere. In this case T^2 has a sequence of eigenvalues $1 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \cdots$ decreasing to zero. For the disc, $\lambda_1 = \lambda_2 = \cdots = 0$; in general, λ_1 is a measure for the angle between L_a^2 and CL_a^2. Here the author obtains the compactness result from more general results of his own. Another result of the author (not proved here) shows the following. Under mild assumptions on $\partial\Omega$, T has finite rank if and only if Ω is a quadrature domain relative to $L_a^2(\Omega)$.

In the end the author touches on a variety of problems related to the topics of his book. We only mention propagation of singularities for elliptic equations and boundary regularity of quadrature domains. It is shown by an example that the singularities of the Schwarz potential in $\mathbf{R}^n$ are best studied from the viewpoint of $\mathbf{C}^n$. In connection with boundary regularity, there is a brief reference to the recent classification by Sakai [6] of all domains in $\mathbf{C}$ with a (generalized) Schwarz function.

In mathematics we are rightly concerned about attracting the (b)right students to our graduate programs. To that end, expositions such as the present one can be of immense value. Shapiro shows with great skill, sense of history, originality, and enthusiasm how a simple concept can grow and play a role in a number of contexts. Thus his booklet would be excellent for an advanced student seminar. Under the guidance of a capable teacher, everyone should have a stimulating experience (including the teacher). Moreover, the students would be introduced to a broad selection of contemporary methods of analysis.

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Viability theory, by Jean-Pierre Aubin. Birkhäuser, Basel, 1991, xxv + 533 pp., $94.50. ISBN 0-8176-3571-8

A variety of problems in applied areas have the following mathematical formulation. The state of a system at time t is described by a vector in $\mathbb{R}^n$, $x(t) = (x_1(t), \dots, x_n(t))$. The evolution of the system is governed by a system of ordinary differential equations

$$\dot{x} = dx/dt = f(x, u(t)), \qquad x(t_0) = x_0, \tag{1}$$

where $u(\ ) = (u_1(\ ), \dots, u_m(\ ))$ is a function with values in $\mathbb{R}^m$ chosen by a controller to achieve a certain objective, such as stabilizing the system, or bringing the state to some preassigned set $\mathscr{T}$, or bringing the state to $\mathscr{T}$ and minimizing some functional $J(\varphi(\ ), u(\ ))$, where $\varphi(\ )$ is a solution of (1) corresponding to the control $u(\ )$. An example of J is

$$J(\varphi(\ ), u(\ )) = g(\varphi(t_f)) + \int_{t_0}^{t_f} f^0(\varphi(s), u(s))\, ds, \tag{2}$$

where f^o is a given function, g is a given function defined on $\mathscr{T}$, t_f is the time at which the state first hits $\mathscr{T}$, and $\varphi(t_f)$ is the point at which the state hits $\mathscr{T}$. This is one form of the optimal control problem. Another example is the "tracking problem": Find $(\varphi(\ ), u(\ ))$ that minimizes $\max\{t_0 \le t \le t_f : |\varphi(t) - \xi(t)|\}$, where $\xi(\ )$ is a prescribed function. In these problems the control $u(\ )$ is usually required to satisfy constraints of the form $u(t) \in \Omega_0$, where Ω_0 is a subset of $\mathbb{R}^m$. Such controls are called "open loop" controls. More generally, the control $u(\ )$ may be required to satisfy constraints of the form $u(t) \in \Omega(\varphi(t))$, where Ω is a mapping from $\mathbb{R}^n$ to subsets of $\mathbb{R}^m$. For many problems Ω_0 and Ω are defined by systems of inequalities. A more demanding, and in applications a more useful, requirement on the control is