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Viability theory, by Jean-Pierre Aubin. Birkhäuser, Basel, 1991, xxv + 533 pp., $94.50. ISBN 0-8176-3571-8

A variety of problems in applied areas have the following mathematical formulation. The state of a system at time t is described by a vector in $\mathbb{R}^n$, $x(t) = (x_1(t), \dots, x_n(t))$. The evolution of the system is governed by a system of ordinary differential equations

$$\dot{x} = dx/dt = f(x, u(t)), \qquad x(t_0) = x_0, \tag{1}$$

where $u(\ ) = (u_1(\ ), \dots, u_m(\ ))$ is a function with values in $\mathbb{R}^m$ chosen by a controller to achieve a certain objective, such as stabilizing the system, or bringing the state to some preassigned set $\mathcal{J}$, or bringing the state to $\mathcal{J}$ and minimizing some functional $J(\varphi(\ ), u(\ ))$, where $\varphi(\ )$ is a solution of (1) corresponding to the control $u(\ )$. An example of J is

$$J(\varphi(\ ), u(\ )) = g(\varphi(t_f)) + \int_{t_0}^{t_f} f^0(\varphi(s), u(s))\, ds, \tag{2}$$

where f^o is a given function, g is a given function defined on $\mathcal{J}$, t_f is the time at which the state first hits $\mathcal{J}$, and $\varphi(t_f)$ is the point at which the state hits $\mathcal{J}$. This is one form of the optimal control problem. Another example is the "tracking problem": Find $(\varphi(\ ), u(\ ))$ that minimizes $\max\{t_0 \le t \le t_f : |\varphi(t) - \xi(t)|\}$, where $\xi(\ )$ is a prescribed function. In these problems the control $u(\ )$ is usually required to satisfy constraints of the form $u(t) \in \Omega_0$, where Ω_0 is a subset of $\mathbb{R}^m$. Such controls are called "open loop" controls. More generally, the control $u(\ )$ may be required to satisfy constraints of the form $u(t) \in \Omega(\varphi(t))$, where Ω is a mapping from $\mathbb{R}^n$ to subsets of $\mathbb{R}^m$. For many problems Ω_0 and Ω are defined by systems of inequalities. A more demanding, and in applications a more useful, requirement on the control is that it be a "feedback control", that is, that the control be dependent on the state. In other words, it is required to find a function $U(\ )$ defined on $\mathbb{R}^n$ with values in $\mathbb{R}^m$, with $U(x) \in \Omega(x)$, and such that if $U(x)$ replaces $u(t)$ in (1), the resulting solution $\varphi(\ )$ and function $u(\ )$ defined by $u(t) = U(\varphi(t))$ achieve the desired objective.

Subsequent to the initial work in the 1960s and early 1970s in the mathematical theory of control, it was noted that in many problems instead of focusing on the control and constraints, it is advantageous to consider the set of permissible velocities or directions, that is, the sets, $F(x) = \{y \in \mathbb{R}^n : y = f(x, z),\ z \in \Omega(x)\}$. The state now evolves according to a "differential inclusion"

$$\dot{x}(t) \in F(x(t)), \qquad x(t_0) = x_0. \tag{3}$$

Note that the differential inclusion incorporates the control constraints and dispenses with the smoothness requirements usually placed on the mapping f. Some regularity requirements on the mapping F are, of course, still required. Under reasonable conditions, we can pass from the differential inclusion solutions of (3) to solutions of control systems (1) by means of "measurable selection theorems". The study of differential inclusions goes back to the 1930s in the work of Marchaud [4, 5] and Zaremba [6], who studied them in a different context and with different terminology.

Also present in many control problems arising in applications are constraints on the state of the form $\psi(t) \in K$, where $\psi(\ )$ is a solution of (3) and K is a preassigned set in $\mathbb{R}^n$. This is a paradigm for the viability problem that is the central problem of the book under review.

Viability Problem. Given a mapping F with domain $\mathrm{Dom}(F) \subset \mathbb{R}^n$ and range the subsets of $\mathbb{R}^n$, find an absolutely continuous function $\psi(\ )$, defined on an interval $[t_0, T]$, such that $\psi(\ )$ is a solution of (3) and such that $\psi(t) \in K \cap \mathrm{Dom}(F)$ for all $t \in [0, T]$.

We now describe the central results of the book. Let X be a finite-dimensional vector space. Let F be a mapping with domain $\mathrm{Dom}(F) \subset X$ and with range the subsets of X. Let K be a subset of $\mathrm{Dom}(F)$. Let I be an interval on the real line. A function $x(\ ) : I \longrightarrow X$ is said to be *viable* in K on I if and only if $x(t) \in K$ for all t in I. The set K is said to be *locally viable* under F if for any initial state x_0 in K there exists a $T > 0$ and a solution $\psi(\ )$ of (3) starting at x_0 and defined in $[0, T]$ which is viable in K. The set K is *globally viable* under F if we can take $T = +\infty$. The subset K is said to be *locally invariant* under F if for any $x_0 \in K$ *all* solutions of (3) are viable in K on some interval. The set K is said to be *invariant* under F if for any $x_0 \in K$ all solutions of (3) are viable in K for *all* $t \geq 0$. In essence, viability theorems state the following. "If F has suitable properties and if at every point $x \in K$, where $K \subset \mathrm{Dom}(F)$, there is a direction $v \in F(x)$ that 'points into K', then K is viable under F."

To state the viability theorems precisely, one must therefore define what is meant by "the existence of a direction $v \in F(x)$ that points into K". This is provided by the contingent cone $T_K(x)$ to K at $x \in K$, first introduced by Bouligand in [3]. A vector v in X belongs to $T_K(x)$ if and only if there exists a sequence of real numbers $h_n > 0$ converging to zero and a sequence v_n in X converging to v such that for every n, $x + h_n v_n \in K$. A subset

$K \subset \mathrm{Dom}(F)$ is said to be a *viability domain* of F if and only if for every $x \in K$, $T_K(x) \cap F(x) \neq \varnothing$. We now paraphrase some of the central results. (Theorems 3.32–3.35).

Theorem. *At each point $x \in Dom(F)$ let F be upper semicontinuous and let $F(x)$ be compact and convex. Let K be a locally compact subset of $Dom(F)$.*

(1) *Then K is locally viable under F if and only if K is a viability domain.*

(2) *If K is a closed subset of $Dom(F)$ and K is a viability domain, then there exists a $T > 0$ and a solution of (3) in $[t_0, T]$ viable in K, where either $T = +\infty$ or $T < \infty$ and $\limsup_{t \longrightarrow T-} \|x(t)\| = +\infty$.*

(3) *If K is closed and F is of linear growth, then $T = +\infty$ and $\|x(t)\|$ and $\|\dot{x}(t)\|$ are at most of exponential growth.*

Of course, there is much more to this book than these theorems. Indeed, the book is a compendium of the state of knowledge about viability at the time of publication and, in a sense, is a revised and expanded version of the 1984 book entitled *Differential inclusions* by the author and A. Cellina [1]. Since an enumeration of the topics treated would amount to a reproduction of the table of contents, we list only those topics that were of interest to this reviewer, whose interest in the book was in potential relevance for optimal control theory and related areas. Those topics are the determination of criteria for a set K to be invariant and locally invariant, the determination of feedback controls that guarantee viability of solutions, functional differential inclusions, and Lyapounov functions.

Mathematically, the book should be accessible to anyone who has had basic graduate courses in modern analysis and functional analysis and who has the motivation and/or patience to master the concepts and proof techniques developed in the author's book with H. Frankowska, *Set valued analyses* [2]. Although many of the arguments rely heavily on these concepts and techniques, the concepts are defined and many proofs of the requisite results from [2] are reproduced here, making the present book essentially self-contained.

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