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CLOSED IDEALS OF THE ALGEBRA OF ABSOLUTELY CONVERGENT TAYLOR SERIES

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Abstract. Let Γ be the unit circle, $A(\Gamma)$ the Wiener algebra of continuous functions whose series of Fourier coefficients are absolutely convergent, and A^+ the subalgebra of $A(\Gamma)$ of functions whose negative coefficients are zero. If I is a closed ideal of A^+, we denote by S_I the greatest common divisor of the inner factors of the nonzero elements of I and by I^A the closed ideal generated by I in $A(\Gamma)$. It was conjectured that the equality $I^A = S_I H^\infty \cap I^A$ holds for every closed ideal I. We exhibit a large class $\mathcal{F}$ of perfect subsets of Γ, including the triadic Cantor set, such that the above equality holds whenever $h(I) \cap \Gamma \in \mathcal{F}$. We also give counterexamples to the conjecture.

1. Introduction

Let D be the open unit disk, let Γ be the unit circle, and let $H^\infty = H^\infty(D)$ be the algebra of bounded holomorphic functions on D. Let A^+ be the algebra of absolutely convergent Taylor series, i.e., the algebra of analytic functions $f\colon D \to \mathbb{C}$ such that $\|f\|_1 = \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!} < +\infty$. Clearly, A^+ is a subalgebra of the disc algebra $\mathcal{U}(D)$ consisting of functions continuous on $\overline{D}$ and holomorphic on D. Also, if we denote by $A(\Gamma)$ the algebra of absolutely convergent Fourier series, equipped with the norm $\|f\|_1 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$, and we identify $f \in A^+$ with $f_{|\Gamma}$, we can write $A^+ = \{f \in A(\Gamma) : \hat{f}(n) = 0 (n < 0)\}$.

Let $I \neq \{0\}$ be a closed ideal of A^+, let S_I be the greatest common divisor of the inner factors of all nonzero elements of I, I^A the closed ideal of $A(\Gamma)$ generated by I, and $h(I) = \{z \in \overline{D}\colon f(z) = 0 (f \in I)\}$. If $E \subseteq \Gamma$ is closed, set $I^+(E) = \{f \in A^+ : f_{|E} \equiv 0\}$. It was proved long ago by Carleson [3] that $I^+(E) = \{0\}$ for certain closed sets E of measure zero. The structure of those closed ideals I of A^+ such that $h(I)$ is finite or countable was described in 1972 by Kahane [12] and Bennett-Gilbert [2]. In this case, $I = I^+(h(I) \cap \Gamma) \cap S_I \cdot H^\infty$. Bennett-Gilbert conjectured in [2] that, in general,

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Conjecture 1. $I = I^A \cap S_I \cdot H^\infty$.

(This conjecture was also quoted by Kahane [12].) Similar conjectures have been verified by Beurling-Rudin [19] for $\mathcal{U}(D)$, by Taylor-Williams [20] for $A^\infty(D) = \{f \in \mathcal{U}(D) : f^{(n)} \in \mathcal{U}(D)\ (n \geq 1)\}$, by Korenblum [16] for $A^p(D) = \{f \in \mathcal{U}(D) : f^{(n)} \in \mathcal{U}(D)\ (n \leq p)\}$, and by Matheson [17] for

$$\Lambda_\alpha(D) = \{f \in \mathcal{U}(D) : \lim \frac{|f(e^{it_1}) - f(e^{it_2})|}{|t_1 - t_2|^\alpha} = 0 \text{ uniformly as } |t_1 - t_2| \to 0)\}.$$

The purpose of this note is to describe recent progress concerning closed ideals of A^+.

The main results can be described as follows:

Theorem 1.1 [6]. *Let p be an integer ≥ 3. If $h(I) \cap \Gamma \subseteq E_{1/p}$, the perfect symmetric set of constant ratio $\frac{1}{p}$, then I satisfies Conjecture* 1.

Theorem 1.2 [7]. *There exists a Kronecker set E and a closed ideal I of A^+ such that $S_I = 1$, $h(I) = E$, which does not satisfy Conjecture* 1.

Results for $L^1(\mathbb{R}^+)$ which concern ideals whose hull is countable and are analogous to the results of Kahane and Bennett-Gilbert have been obtained by Nyman [18] and Gurarii [9, 10]. By using transfer methods due to Hedenmalm [11], El Fallah [4] has proved certain versions of Theorem 1.1 and Theorem 1.2 for $L^1(\mathbb{R}^+)$.

2. Division ideals

If $I \neq \{0\}$ is a closed ideal of A^+ and if $f \in A^+$, we denote by $I(f) = \{g \in A^+ : fg \in I\}$ the *division ideal* associated with f. We will say that a closed subset $E \subseteq \Gamma$ is a Carleson set if

$$\int_{-\pi}^{\pi} \log\left(\frac{2}{\operatorname{dist}(e^{it}, E)}\right) dt < \infty.$$

This condition is necessary and sufficient for the existence of a nonzero $f \in \Lambda_\alpha$ vanishing on E. It is also necessary and sufficient for the existence of an outer $f \in A^\infty(D)$ such that f vanishes exactly on E and $f^{(n)}_{|E} = 0$ $(n \geq 1)$; see [3, 20].

Some improvements of the methods used to discuss closed ideals of $A^\infty(D)$ and $A^+(D)$ lead to the following result (one must circumvent the fact that the outer part of an element of A^+ does not necessarily belong to A^+):

Theorem 2.1. (i) *If $f \in A^+ \cap S_I H^\infty$, then $h(I(f)) \subseteq h(I) \cap \Gamma$. Also, the positive singular measure which defines the inner factor of $I(f)$ is nonatomic and vanishes on all Carleson sets.*

(ii) *If, further, f vanishes on $h(I) \cap \Gamma$, then $h(I(f))$ is a perfect subset of Γ.*

We note that Theorem 2.1(ii) contains the results of Kahane and Bennett-Gilbert, for, if $h(I)$ is countable, it implies that $h(I(f)) = \varnothing$.

The results and methods of [1, 17, 20] lead, also, to the following information:

Theorem 2.2. *Let E be a Carleson set, and denote by $J_0^+(E)$ the closure in A^+ of the set of elements of $A^\infty(D)$ vanishing on E with all their derivatives. Then*:

(i) *$J_0^+(E) \subseteq I$ for every closed ideal I of A^+ such that $S_I = 1$ and $h(I) \subseteq E$.*

(ii) *If $\alpha > \frac{1}{2}$ and $f \in \Lambda_\alpha(D) \cap I^+(E)$, then $f \in J_0^+(E)$.*

3. When the conjecture works

The following lemma, related to the Katznelson-Tzafriri theorem for contractions [5, 14], is the key to our positive results concerning the Bennett-Gilbert conjecture. The $w^\star$ topology discussed below is defined by considering A^+ as the dual of c_0.

Lemma 3.1. *Let I be a closed ideal of A^+, and let $f \in I^A \cap A^+$. Then $I(f)$ is $w^\star$-closed.*

Using the results of §2, we obtain:

Theorem 3.2 [6]. *Let $E \subseteq \Gamma$ be a Carleson set. If $J_0^+(E)$ is $w^\star$-dense in A^+, then $I = I^A \cap S_I H^\infty$ for every closed ideal of A^+ such that the perfect part of $h(I) \cap \Gamma$ is contained in E.*

Thus, Theorem 1.1 follows from the fact that $J_0^+(E_{1/p})$ is indeed $w^\star$-dense in A^+.

4. When the conjecture fails

If K is a Helson set, then $I^+(K)$ is $w^\star$-dense in A^+ (this observation is an extension of [8, Theorem 4.5.2]). Also, if K is a Kronecker set and a Carleson set, then $J_0^+(K)$ has no inner factor; so the closed ideal generated by $J_0^+(K)$ consists of all functions of $A(\Gamma)$ vanishing on K, since Kronecker sets satisfy synthesis [21]. So, if the ideal $J_0^+(K)$ satisfies the Bennett-Gilbert conjecture, we must have $J_0^+(K) = I^+(K)$. A construction which has some relation with Kaufman's construction of a Helson set of multiplicity [15] gives the following result:

Theorem 4.1 [7]. *Let E be a set of multiplicity. Then there exists a nonzero distribution μ whose support is a Kronecker subset of E such that $\hat{\mu}(n) \to 0$ as $n \to -\infty$.*

Theorem 1.2 follows from Theorem 4.1 applied to a Carleson set of multiplicity (for example, the perfect set E_ξ when $\frac{1}{\xi}$ is not a Pisot number). In this case, if K is the support of the distribution given by Theorem 4.1, we have $J_0^+(K)$ properly contained in $I^+(K)$, and it is even possible to show that there are $2^{\aleph_0}$ distinct closed ideals between $J_0^+(K)$ and $I^+(K)$, ideals which would be equal if the Bennett-Gilbert conjecture were true.

5. Applications

The positive results about the Bennett-Gilbert conjecture give "strong uniqueness properties" of some closed subsets of Γ. For example, it follows from Theorem 2.1 that any distribution S supported by $E_{1/p}$ such that $\widehat{S}(n) \to 0$ as $n \to \infty$ must be the zero distribution. Some stronger results involving hyperdistributions can be found in [7]. We present here an application of Theorem 1.1 to operator theory (an extension of the Beurling-Pollard method [13, p. 61] is involved in the proof).

Theorem 5.1. *Let T be a contraction on a Banach space. If $\operatorname{Sp} T \subseteq E_{1/p}$ and if*

$$\limsup_{n\to\infty} \frac{\log^+ \|T^{-n}\|}{n^\alpha} < +\infty \quad \text{where } \alpha < \frac{\log p - \log 2}{2\log p - \log 2},$$

then $\sup_{n\geq 1} \|T^{-n}\| < +\infty$.

If we add to the hypotheses of Theorem 5.1 the assumption that $\operatorname{Sp} T$ is a Dirichlet set, we can conclude that T is an isometry. Analogous results hold for all E_ξ with $\xi \in (0, \frac{1}{2})$ if we consider only Hilbert spaces. On the other hand, if E is a set of multiplicity (for example, $E = E_\xi$ when $\frac{1}{\xi}$ is not a Pisot number), then there exist contractions T such that $\operatorname{Sp} T \subseteq E$ and $\|T^{-n}\|$ goes to infinity arbitrarily slowly as n goes to infinity.

Zarrabi [22] proved that if $E \subseteq \Gamma$ is countable, every contraction on a Banach space X such that $\operatorname{Sp} T \subseteq X$ and $\log^+ \|T^{-n}\|/n^{1/2} \to 0$ as $n \to \infty$ is an isometry; but this property holds exclusively for countable sets, even if we suppose that X is a Hilbert space.

It follows from unpublished computations by M. Zarrabi, M. Rajoelina, and the first author that the constant $\frac{\log p - \log 2}{2\log p - \log 2}$ in Theorem 5.1 is the best possible.

References

1. A. Atzmon, *Operators which are annihilated by analytic functions and invariant subspaces*, Acta. Math. **144** (1980), 27–63.
2. C. Bennett and J. E. Gilbert, *Homogeneous algebras on the circle: I-ideals of analytic functions*, Ann. Inst. Fourier Grenoble **22** (1972), 1–19.
3. L. Carleson, *Sets of uniqueness of functions regular in the unit circle*, Acta Math. **87** (1952), 325–345.
4. O. El Fallah, *Idéaux fermés de $L^1(\mathbb{R}^+)$*, Math. Scand. (1) **72** (1993), 120–130.
5. J. Esterle, E. Strouse, and F. Zouakia, *Theorems of Katznelson-Tzafriri type for contractions*, J. Funct. Anal. **94** (1990), 273–287.
6. ———, *Closed ideals of A^+ and the Cantor set*, J. Reine Angew. Math. (to appear).
7. J. Esterle, *Distributions on Kronecker sets, strong forms of uniqueness, and closed ideals of A^+*, J. Reine Angew. Math. (to appear).
8. C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
9. V. P. Gurarii, *Spectral synthesis of bounded functions on the half axis*, Funct. Anal. Prilozhen **4** (1969), 34–48.
10. ———, *Harmonic analysis in spaces with weight*, Trans. Moscow Math. Soc. **35** (1979), 21–75.
11. H. Hedenmalm, *A comparison between the closed ideals in l^1_ω and L^1_ω*, Math. Scand. **58** (1986), 275–300.
12. J. P. Kahane, *Idéaux fermés dans certaines algèbres de fonctions analytiques*, Actes Table Ronde Int. C. N. R. S. Montpellier, Lecture Notes in Math., vol. 336, Springer-Verlag, Berlin, Heidelberg, and New York, 1973, pp. 5–14.
13. ———, *Series de Fourier absolument convergentes*, Ergeb. Math. Grenzgeb. (3), vol. 50, Springer-Verlag, Berlin, Heidelberg, and New York, 1970.
14. Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. **68** (1986), 313–328.
15. R. Kaufman, *M-sets and distributions*, Asterisque **5** (1973), 225–230.
16. B. I. Korenblum, *Closed ideals in the ring A^n*, Funct. Anal. Appl. **6** (1972), 203–214.
17. A. L. Matheson, *Closed ideals in rings of analytic functions satisfying a lipschitz condition*, Lecture Notes in Math., vol. 604, Springer-Verlag, Berlin, Heidelberg, and New York, 1976, pp. 67–72.
18. B. Nyman, *On the one dimensional translation group and semigroup in certain function spaces*, Thesis, Uppsala, 1950.
19. W. Rudin, *The closed ideals in an algebra of analytic functions*, Canad. J. Math. **9** (1957), 426–434.
20. B. A. Taylor and D. L. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Canad. J. Math. **22** (1970), 1266–1283.

21. N. Varopoulos, *Sur les ensembles parfaits et les series trigonometriques*, C. R. Acad. Sci. Paris Sér. I. Math. **260** (1965), 3831–3834.
22. M. Zarrabi, *Contractions à spectre denombrable et propriétés d'unicité forte des fermés denombrables du cercle*, Ann. Inst. Fourier (1) **43** (1993), 251–263.

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