APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 1, July 1994, Pages 119-125

BOOK REVIEW

Theory of function spaces II, by Hans Triebel. Birkhäuser Verlag, Basel, 1992, viii+370 pp., \$117.00. ISBN 3-7643-2639-5

1. Littlewood-Paley-Stein theory

The theory of function spaces appears at first to be a disconnected subject, because of the variety of spaces and the different considerations involved in their definitions. There are the Lebesgue spaces L^p (defined by size); the Lipschitz spaces Λ^α (smoothness); the Sobolev spaces L^p_k, $k \in \mathbb{N}$ (size and smoothness); their generalizations, the fractional potential spaces L^p_α, $\alpha \in \mathbb{R}$ (defined via the Fourier transform); the generalized Lipschitz spaces $\Lambda^{p,q}_\alpha$ (size and smoothness in a mixed norm); the real-variable Hardy spaces H^p (boundary values of generalized Cauchy-Riemann systems, nontangential maximal function, atomic decomposition); and BMO (bounded mean oscillation).

Nevertheless, several approaches lead to a unified viewpoint on these spaces, for example, approximation theory or interpolation theory. One of the most successful approaches, and the one taken in Triebel's book, is Littlewood-Paley-Stein (LPS) theory. This arose first in the 1930s in connection with Fourier series and analytic functions on the unit disk (see [Z, Chapters 14 and 15]). Its extension to $\mathbb{R}^n$ and beyond began with [St1] in 1958. A good historical account is [St3], while the key reference for over twenty years has been [St2], much of which was updated in [St4].

The classical formulation of LPS theory on $\mathbb{R}^n$ is in terms of the Dirichlet problem on $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$. For a boundary value function $f \in L^p(\mathbb{R}^n)$, $1 \le p \le +\infty$, the various auxiliary Littlewood-Paley functions $g(f)$, $g_x(f)$, $g_1(f)$, $S(f)$, and $g^*_\lambda(f)$ are formed using the harmonic extension $P_t * f(x)$ (the solution to the Dirichlet problem), where $P_t(x)$ is the Poisson kernel. For example,

$$g_1(f)(x) = \left(\int_0^\infty \left| t \frac{\partial}{\partial t} P_t * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} . \tag{1}$$

To make a long story short, the following norm equivalences hold for $1 < p < +\infty$:

$$\|f\|_{L^p} \approx \|g(f)\|_{L^p} \approx \|g_x(f)\|_{L^p} \approx \|g_1(f)\|_{L^p} \approx \|S(f)\|_{L^p} \approx \|g^*_\lambda(f)\|_{L^p} \tag{2}$$

(for $\lambda > 2/p$ in the case of g^*_λ). This leads to some very precise results, such as a local a.e. characterization of nontangential convergence ([St2, p. 206]). Thus we can view (2) as an application of function spaces to the study of the Dirichlet problem.

However, LPS theory as a subject in itself starts when we take the somewhat perverse opposite view: (2) is an application of the Dirichlet problem to provide alternate norms for use in studying L^p. The idea of replacing the simple L^p norm by a complicated norm like $\|g_1(f)\|_{L^p}$, defined using the usual L^p norm anyway, appears crazy and, in fact, horrified a colleague of mine. But certain types of structure may be exhibited only via the alternate norm. A prototype of such a case is Stein's proof [St2, p. 96] that a Fourier multiplier operator T_m satisfying a Hörmander estimate actually satisfies the pointwise inequality $g_1(T_m f)(x) \le c g_\lambda^*(f)(x)$ and hence is L^p bounded by (2). Nowadays, LPS techniques are commonplace; one interesting example among many is the fractional derivative chain rule estimate in [CW].

The LPS square function $S(f)$ played a key role in [BGS] and [FS], which touched off the rapid development in the 1970s of the $H^p(\mathbb{R}^n)$ theory of Stein and Weiss. Among other things, these papers show that the H^p spaces and BMO have LPS characterizations (involving Carleson measures in the case of BMO). Thus LPS theory provides one of the senses in which the spaces $H^p - L^p - \mathrm{BMO}$ form a unified scale.

In another direction, around 1960, Besov systematically generalized the Lipschitz spaces. For example, for $0 < \alpha < 1$ and $1 \le p,\, q \le +\infty$, define the Besov space B^s_{pq} (or $\Lambda^{p,q}_s$) by the norm

$$\|f\|_{B^s_{pq}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^n} \left(|t|^{-s} \|f(x+t) - f(x)\|_{L^p(dx)} \right)^q \frac{dt}{|t|^n} \right)^{1/q} . \tag{3}$$

Taibleson [Ta] proved that B^s_{pq} also has an LPS characterization:

$$\|f\|_{B^s_{pq}} \approx \|f\|_{L^p} + \left(\int_0^\infty \left(t^{1-s} \left\| \frac{\partial}{\partial t} P_t * f(x) \right\|_{L^p(dx)} \right)^q \frac{dt}{t} \right)^{1/q} . \tag{4}$$

(B^s_{pq} should probably be called a Besov-Lipschitz-Taibleson space, except for the resulting abbreviation.) If we take $s = 0$ and $q = 2$, (4) is similar to $\|g_1(f)\|_{L^p}$ except for the order with which the $L^p(\mathbb{R}^n)$ and $L^2((0,\infty), dt/t)$ norms are taken. Since L^p and Λ^α do not seem closely related, this is surprising. Here is a heuristic explanation of (4). We can write $t\frac{\partial}{\partial t} P_t * f(x) = \varphi_t * f(x)$, where $\varphi_t(x) = t^{-n}\varphi(x/t)$, for a certain φ satisfying $\int \varphi = 0$. Letting δ_p be the point mass at p, we can also write $f(x+t) - f(x) = (\delta_{-t} - \delta_0) * f(x)$; and $\delta_{-t} - \delta_0$ is like an $\mathbb{R}^n$ dilation of the mean-zero kernel $\delta_e - \delta_0$, where $e = (-1, 0, \ldots, 0)$. The modern perspective, which explains many of the equivalent characterizations of various function spaces, is that one mean-zero kernel is generally as good as another.

The Besov spaces may appear quite technical, but they arise naturally in many contexts. In boundary value partial differential equations, $B^{\alpha-1/p}_{pp}(\mathbb{R}^n)$ is the space obtained when one restricts $L^p_\alpha(\mathbb{R}^{n+1})$ to $\mathbb{R}^n$ (if $\alpha > 1/p$). Besov spaces often appear in sharp versions of Fourier multiplier theorems, e.g., in [BS, Se], and Taibleson's characterization of Fourier multipliers bounded on Λ^α [Ta]. The fractal dimension of a graph is characterized via Besov spaces in [DeJ]. In statistics Donoho and Johnstone [DoJ] found the unit ball of a Besov space more natural than the unit ball of a Sobolev space for minimax analysis of estimators.

The parameter s in (3) reflects the action of a Bessel potential. It can be incorporated similarly in (1) and (2) to give an LPS characterization of the Sobolev and potential spaces. Thus the LPS approach describes all of the spaces noted in the first paragraph, except for certain problem endpoint spaces like L^1 and L^∞.

2. The European school

So far our notation has been that of the Stein school. Another school developed in the 1960s and 1970s in Sweden, Eastern Europe, and the former Soviet Union, which reached a similar unification of function space theory by a different path. Motivated probably by methods of Hörmander in studying partial differential equations, they used a Fourier transform approach. Pick Schwartz functions Φ and φ on $\mathbb{R}^n$ satisfying $\operatorname{supp} \widehat{\Phi} \subseteq \{\xi \colon |\xi| \le 2\}$, $\operatorname{supp} \hat{\varphi} \subseteq \{\zeta \colon 1/2 \le |\xi| \le 2\}$, and the nondegeneracy condition $|\widehat{\Phi}(\xi)|$, $|\hat{\varphi}(\xi)| \ge c > 0$ on slightly smaller regions. For $j \in \mathbb{Z}$, let $\varphi_j(x) = 2^{jn}\varphi(2^j x)$. Peetre proved [P1] in 1967 that

$$\|f\|_{B^s_{pq}} \approx \|\Phi * f\|_{L^p} + \left(\sum_{j=1}^{\infty} (2^{js} \|\varphi_j * f\|_{L^p})^q \right)^{1/q} . \tag{5}$$

He also considered the homogeneous Besov spaces $\dot{B}^s_{pq}$ defined by the norm $(\sum_{j \in \mathbb{Z}} (2^{js} \|\varphi_j * f\|_{L^p})^q)^{1/q}$.

Independently, Triebel [Tr1] and Lizorkin [L] introduced F^s_{pq} (the Triebel-Lizorkin spaces), defined originally for $1 \le p < +\infty$, $1 \le q \le +\infty$ by the norm

$$\|f\|_{F^s_{pq}} = \|\Phi * f\|_{L^p} + \left\| \left(\sum_{j=1}^{\infty} (2^{js} |\varphi_j * f|)^q \right)^{1/q} \right\|_{L^p} , \tag{6}$$

and the homogeneous analogue $\dot{F}^s_{pq}$ with norm $\|(\sum_{j \in \mathbb{Z}} (2^{js} |\varphi_j * f|)^q)^{1/q}\|_{L^p}$. Motivated by H^p theory, especially [FS], Peetre extended these definitions to $0 < p$, $q \le 1$ (see [P2] for a nice account).

If we write φ_t for $t\partial P_t/\partial t$ as above and note that $\int_{2^j}^{2^{j+1}} dt/t$ is independent of $j \in \mathbb{Z}$, we see that (5) is analogous to (4) and (6) is analogous to $\|g_1(f)\|_{L^p}$. Then by the LPS results noted earlier and the expectation that different kernels give the same results, the following equivalences seem natural: $L^p \approx F^0_{p2} \approx \dot{F}^0_{p2}$ if $1 < p < +\infty$, $H^p \approx \dot{F}^0_{p2}$ if $0 < p \le 1$, $L^p_\alpha \approx F^\alpha_{p2}$ if $\alpha > 0$ and $1 < p < +\infty$, and $\Lambda_\alpha \approx B^\alpha_{\infty\infty}$ if $\alpha > 0$. With a natural Carleson norm definition of $\dot{F}^s_{\infty q}$ (see [FJ]) we even have BMO $\approx \dot{F}^0_{\infty 2}$. So the Besov and Triebel-Lizorkin scales systematically incorporate the full range of spaces that are unified by LPS theory. Although this was not the original motivation, we can regard (5) and (6) as turning the LPS characterizations into the definitions of the spaces.

The European formulation has certain advantages. For example, the theory of functions of exponential type can be applied to each $\varphi_j * f$ (since $\operatorname{supp} \hat{\varphi}_j$ is compact). With this, the Sobolev embedding theorem can be given a very simple proof and can even be sharpened within the B^s_{pq} and F^s_{pq} scales (see [Tr3, p. 129]). Many apparently distinct results for H^p, L^p, Λ_α, etc., can be shown to be essentially the same by giving them a single proof in the B and F space notation. Beyond this, the European school has given rise to new insights leading to new results, such as those in [Se]. Finally, the introduction of discrete expressions can be seen as one step in the direction of modern wavelet theory.

3. The connection with wavelets

A fundamental difficulty in using function space theory in Fourier analysis is that for $p \neq 2$, L^p is not characterized by the size of the Fourier transform (L^2 is so characterized, by Plancherel's theorem). This is nicely described by C. Fefferman in his 1974 ICM lecture [F]:

> Take a function $f(x) \sim \sum_{-\infty}^{+\infty} a_k e^{ikx}$ belonging to L^p $(p < 2)$ but not to L^2, and modify its Fourier series by writing $g(x) \sim \sum_{-\infty}^{+\infty} \pm a_k e^{ikx}$ with each $\pm$ sign picked independently by flipping a coin. Then with probability one, g does not belong to L^p (or even to L^1) but is merely a distribution with nasty singularities.

See [Z, Chapter 5] for these results. Fefferman returns to this issue at the end of his lecture:

> Perhaps in dealing with the Fourier transform in $\mathbb{R}^n$, we must abandon our fixation on Lebesgue measure and search for new quantities (defined possibly in terms of coverings by thin rectangles) to express the size or importance of a set of points. This is easier said than done.

For those of us looking for simpler projects, the alternative to changing the function spaces is to change the transform. In fact, the wavelet transform does match the standard function spaces in the desired sense. In 1985 Meyer [M1] constructed a Schwartz function ψ on $\mathbb{R}$ such that the set $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$, where $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, is an orthonormal basis for $L^2(\mathbb{R})$. The map $f \to \{\langle f, \psi_{jk}\rangle\}$ is called the wavelet transform, which is inverted by the wavelet identity

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk}\,. \tag{7}$$

For $1 < p < +\infty$, L^p is precisely characterized by the magnitudes of the wavelet coefficients:

$$\|f\|_{L^p} \approx \left\| \left(\sum_{j,k} (2^{j/2} |\langle f, \psi_{jk}\rangle| \chi_{[2^{-j}k, 2^{-j}(k+1)]})^2 \right)^{1/2} \right\|_{L^p} . \tag{8}$$

There are similar characterizations of H^p, Λ_α, L^p_α, $\Lambda^{p,q}_\alpha$, and BMO, and an extension to n dimensions, in [M2]. In fact, the whole Besov and Triebel-Lizorkin scales can be characterized this way, as in [FJW, Theorem 7.10].

By comparison to the Fourier case, (8) seems miraculous. But (7) and (8) have their origin in LPS theory and an identity called the Calderón formula, which goes back to [C] in 1964. One form of this identity is as follows. Let $\varphi_t = t\partial P_t/\partial t$ as above, and pick $\psi \in \mathcal{S}(\mathbb{R}^n)$ to be real and radial and to satisfy $\int \psi = 0$ and $\int_0^\infty \hat{\psi}(t\eta)e^{-t}\,dt = -1$ for $\eta = (1, 0, \dots, 0)$. Then by Fourier inversion we obtain

$$f(x) = \int_0^\infty \int_{\mathbb{R}^n} \varphi_t * f(y)\psi_t(x-y)\,dy\,\frac{dt}{t} \tag{9}$$

(see, e.g., [FJW, p. 4]). The European version, which goes way back also, is similar: $f = \sum_{j\in\mathbb{Z}} \varphi_j * \varphi_j * f$ if $\sum_{j\in\mathbb{Z}} (\hat{\varphi}_j)^2 = 1$ a.e. We regard $\varphi_t * f(y)$ as the coefficient of $\psi_t(x-y)$ in the continuous expansion (9), which is analogous to (7) (especially since $\langle f, \psi_{jk}\rangle = f * \psi_j(2^{-j}k)$ for $\psi_j(x) = 2^{j/2}\overline{\psi(-2^j x)}$). Now by LPS theory, L^p is characterized by the magnitude of $\varphi_t * f(y)$ via $\|g_1(f)\|_{L^p} \approx \|f\|_{L^p}$ and similarly for the other spaces above. So (8) is analogous to (2). This also helps us understand the right side of (8): it is the L^p norm of a discrete Littlewood-Paley expression for the sequence $\{\langle f, \psi_{jk}\rangle\}$. (See [FJ] for a systematic extension of this notion.) The fact that the equivalent norms come down to pure size estimates is probably the main reason for the success of LPS theory (cf. [CMS]).

For function space theorists, wavelet theory can be regarded as a particularly simple and convenient formulation of LPS theory. Of course, for many applications of wavelets, especially in signal compression, the discrete nature of (7) and (8) is essential. Also, the many other predecessors of wavelets in mathematics, engineering, and physics should not be slighted. Still, one of the reasons for the current excitement over wavelets is that the mathematically sophisticated LPS theory has passed a certain threshold of simplicity in its wavelet formulation that makes it accessible to a very general audience.

4. Triebel's book

Despite the title, *Theory of function spaces* II is Triebel's third major text on function spaces. The other two are [Tr2] and [Tr3]. These were systematic and thorough, and hence very useful as reference works, but were somewhat technical. The new book is much more informal, readable, and intuitive. It is more limited in scope: in the author's words (p. v), "However, those topics where we have nothing new to say will not again be treated in detail; we shall refer to [Tr3] or other relevant sources." The topics presented in the new book are mainly of two types: first, those for which there is now a simpler approach than in [Tr3]; second, those needed for the main new material in the book, the generalization of the theory to Riemannian manifolds.

Chapter 1 and the rest of the book are independent of one another. Chapter 1 is a self-contained expository introduction and historical survey, without proofs, of the development of the subject. It is accessible and interesting to nonspecialists as well as specialists. Triebel does a nice job of showing how all the different approaches eventually led to the same set of function spaces.

The main program, with full proofs, starts in Chapter 2, which presents the basic theory of the Besov and Triebel-Lizorkin spaces. The terminology is that of the European school. The main new device for simplifying proofs, the atomic decomposition, is discussed in Chapter 3. The key theorems on pointwise multipliers, diffeomorphisms, traces, and extensions are presented in Chapter 4. In Chapter 5 function spaces on domains in $\mathbb{R}^n$ are defined, originally by restriction from $\mathbb{R}^n$, but eventually by an equivalent intrinsic definition. Results on the B^s_{pq} and F^s_{pq} boundedness of pseudodifferential operators are given in Chapter 6, with proofs based on the atomic decomposition.

The book concludes with Chapter 7, where the generalization of these function spaces to Riemannian manifolds (of positive injectivity radius and bounded geometry) is given. The original definition is based on patching together thedefinition inherited from local charts. The results of Chapter 4 are used to show independence of the choice of charts. The main result gives an equivalent intrinsically defined characterization.

Theory of function spaces II is a valuable and readable book. It is of interest to anyone desiring a general overview of function space theory and especially to someone wanting to consider the extension of this theory to Riemannian manifolds.

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