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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 1, July 1994

0273-0979/94 $1.00 + $.25 per page

The commutant lifting approach to interpolation problems, by Ciprian Foias and Arthur E. Frazho. Birkhäuser, Basel, 1990, 6331 pp., $129.00. ISBN 3-7643-2461-9

1. INTRODUCTION

The literature on interpolation theory is vast and rapidly growing vaster. Part of this growth is due to the fact that the subject is genuinely rich and lends itself to many different approaches and applications. Since 1989 alone, at least five books have been published: Ball, Gohberg, and Rodman [BGR]; Bakonyi and Constantinescu [BC]; Dubovoj, Fritzsche, and Kirstein [DFK]; Dym [D1]; and the present volume by Frazho and Foias [FF]. To this list one should add the earlier monographs of Akhiezer [Ak], Katsnelson [K1], Krein and Nudelman [KN], and Rosenblum and Rovnyak [RR], which contain much relevant material, plus hundreds of journal articles and conference reports and numerous Ph.D. theses. The book under review focuses on the method of commutant lifting which was introduced by Sarason [Sa] in 1967 to solve a pair of classical interpolation problems for scalar-valued analytic functions in the disc. Shortly thereafter, the basic lifting theorem was extended to an abstract operator setting by Nagy and Foias [SNF1, SNF2]. This paved the way for applications to interpolation problems for matrix- and operator-valued functions, and that is what this book is largely about.

To say more, it is convenient to introduce some notation. The symbols $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$ will denote the complex numbers, the open unit disc, and the unit circle, as usual; $\mathbb{C}^{p\times q}$ designates the set of $p\times q$ matrices with entries in $\mathbb{C}$, and $\mathbb{C}^p$ is short for $\mathbb{C}^{p\times 1}$. Similarly, $L_2^k(\mathbb{T})$ and $H_2^k(\mathbb{T})$ alias $H_2^k(\mathbb{D})$ will denote the set of $k\times 1$ vector-valued functions with entries in $L_2(\mathbb{T})$ and the Hardy space $H_2(\mathbb{D})$, respectively, with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})^* f(e^{i\theta})\, d\theta .$$

Here and elsewhere, the superscript $*$ denotes the Hermitian transpose of the indicated vector function and the adjoint (with respect to the appropriate inner product) when applied to an operator. The symbol $\underline{\underline{p}}$ stands for the orthogonal projection of $L_2^k(\mathbb{T})$ onto $H_2^k(\mathbb{T})$ regardless of the size of the integer k, and $\underline{\underline{q}}' = I - \underline{\underline{p}}$. The set of $p \times q$ matrix-valued functions which are both analytic and contractive in $\mathbb{D}$ will be denoted by $\mathscr{S}^{p\times q}(\mathbb{D})$. Finally, $\mathscr{B}(\mathscr{H}, \mathscr{K})$ stands for the bounded linear operators from the Hilbert space $\mathscr{H}$ into the Hilbert space $\mathscr{K}$, $\mathscr{B}(\mathscr{H})$ is short for $\mathscr{B}(\mathscr{H}, \mathscr{H})$, and $\sigma(A)$ designates the spectrum of an operator A.

We shall also make extensive use of the kernel function

$$\rho_\omega(\lambda) = 1 - \lambda\omega^*;$$

the operator R_α which is defined by the rule

$$(R_\alpha f)(\lambda) = \frac{f(\lambda) - f(\alpha)}{\lambda - \alpha},$$

wherever it is meaningful; and the signature matrix

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

The interpolation problems considered in this book fall roughly into two categories. The first fits into the general framework of what the reviewer would term the OBIP (one-sided basic interpolation problem); the second is the Nehari problem. These two problems will be formulated below in the setting of matrix-valued functions. To ease the discussion, however, we begin with a concrete example of the OBIP: the NP (Nevanlinna-Pick) problem.

2. The NP problem

The data for (the one-sided tangential version of) this problem formulated for $\mathbb{D}$ is a given set of points $\omega_1, \dots, \omega_n$ in $\mathbb{D}$ and two sets of vectors $\xi_1, \dots, \xi_n$ in $\mathbb{C}^p$ and $\eta_1, \dots, \eta_n$ in $\mathbb{C}^q$. It is presumed that the functions ξ_j/ρ_{ω_j}, $j = 1, \dots, n$, are linearly independent in $H_2^p(\mathbb{D})$; this is automatic if the points $\omega_1, \dots, \omega_n$ are distinct. The first objective is to find conditions on the data which will guarantee the existence of a $p \times q$ matrix-valued function S in the Schur class $\mathscr{S}^{p\times q}(\mathbb{D})$ such that

$$\xi_j^* S(\omega_j) = \eta_j^* \tag{1}$$

for $j = 1, \dots, n$. The second objective is to find a description of all such S when the conditions for existence are met.

Let $g_j = \xi_j/\rho_{\omega_j}$ and $h_j = \eta_j/\rho_{\omega_j}$ for $j = 1, \dots, n$. Then with the help of the evaluation

$$\underline{\underline{p}}F^* \frac{\xi}{\rho_\omega} = F(\omega)^* \frac{\xi}{\rho_\omega}, \tag{2}$$

which is valid for every choice of $\omega \in \mathbb{D}$, $\xi \in \mathbb{C}^p$, and $F \in H_\infty^{p\times q}(\mathbb{D})$, it is readily seen that (1) holds for some $S \in \mathscr{S}^{p\times q}(\mathbb{D})$ if and only if $\underline{\underline{p}}S^* g_j = h_j$ for $j = 1, \dots, n$. The OBIP is formulated in terms of this last condition but for more general choices of g_j and h_j.

3. The OBIP

Let $F = [f_1 \cdots f_n]$ be an $m \times n$ matrix-valued function with columns

$$f_j = \begin{bmatrix} g_j \\ h_j \end{bmatrix}, \qquad j = 1, \ldots, n,$$

such that

(1) $g_1, \ldots, g_n$ are linear independent in $H_2^p(\mathbb{D})$, and
(2) $h_1, \ldots, h_n$ belong to $H_2^q(\mathbb{D})$.

The problem is twofold:

(1) Formulate necessary and sufficient conditions on the data $f_1, \ldots, f_n$ for the existence of an $S \in \mathscr{S}^{p\times q}(\mathbb{D})$ such that

$$\underset{=}{p}S^* g_j = h_j \quad \text{for } j = 1, \ldots, n. \tag{3}$$

(2) Describe the set of all such "solutions" S when the conditions for existence are met.

The assumption that $g_j \in H_2^p(\mathbb{D})$ and $h_j \in H_2^q(\mathbb{D})$ can be relaxed (it is really the requirement that $S^* g_j - h_j \in H_2^q(\mathbb{D})^\perp$ which is important; see [D1, Theorem 2.7]), but the present formulation is convenient for purposes of exposition and covers many cases of interest.

4. The matrix Nehari problem

Let $\gamma_1, \gamma_2, \ldots,$ be a given set of $p \times q$ matrices. Again the problem is twofold:

(1) Formulate necessary and sufficient conditions for the existence of $G \in L_\infty^{p\times q}(\mathbb{T})$ with $p \times q$ matrix Fourier coefficients

$$G_t = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\theta}) e^{-it\theta}\, d\theta$$

such that

$$G_{-t} = \gamma_t \quad \text{for } t = 1, 2, \ldots$$

and

$$\|G\|_\infty \le 1.$$

(2) Describe the set of all such "solutions" G when the conditions in (1) are met.

5. On solving the OBIP problem

If S is a solution, then clearly

$$\left\| \sum_{j=1}^n x_j h_j \right\|^2 = \left\| \underset{=}{p}S^* \sum_{j=1}^n x_j g_j \right\|^2 \le \left\| \sum_{j=1}^n x_j g_j \right\|^2$$

for every choice of complex coefficients $x_1, \ldots, x_n$. Therefore, the $n \times n$ matrix P with ij entry

$$P_{ij} = \langle g_j, g_i \rangle - \langle h_j, h_i \rangle = \langle J f_j, f_i \rangle, \tag{4}$$

for $i, j = 1, \ldots, n$, must be positive semidefinite: $P \ge 0$.

In general, however, this condition is not sufficient unless an additional constraint is imposed on the data. One convenient way to do this is in terms of the space

$$\mathscr{M} = \{Fu : u \in \mathbb{C}^n\}. \tag{5}$$

The constraint is that $\mathscr{M}$ be R_α invariant for some point $\alpha \in \mathbb{D}$. This is equivalent to restricting $F(\lambda)$ to be of the special form

$$F(\lambda) = C(I_n - \lambda A)^{-1} \tag{6}$$

for some choice of $C \in \mathbb{C}^{m\times n}$ and $A \in \mathbb{C}^{n\times n}$, with $\sigma(A) \subset \mathbb{D}$. Under this restriction, the condition $P \geq 0$ is also sufficient for the existence of a solution to the OBIP problem. In fact, if $F(\lambda)$ is of the form (6) and $P > 0$, then there exists a rational $m \times m$ matrix-valued J inner function (of McMillan degree n) $\Theta(\lambda)$ which is analytic in $\overline{\mathbb{D}}$ such that

$$\underset{=}{p}\Theta^* J f_j = 0 \tag{7}$$

for $j = 1, \ldots, n$. Therefore, upon writing Θ in the block form

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

which is conformal with J (i.e., Θ_{11} is $p\times p$, Θ_{12} is $p\times q$, etc.), it is readily seen that

$$\underset{=}{p}\Theta_{11}^* g_j - \underset{=}{p}\Theta_{21}^* h_j = 0 \tag{8}$$

and

$$\underset{=}{p}\Theta_{12}^* g_j - \underset{=}{p}\Theta_{22}^* h_j = 0 \tag{9}$$

for $j = 1, \ldots, n$. Since Θ_{22} is invertible in $\mathbb{D}$ and

$$S_0 := \Theta_{12}\Theta_{22}^{-1}$$

belongs to $\mathscr{S}^{p\times q}(\mathbb{D})$ for such Θ (see, e.g., [D1, Chapter 3]), it follows from (9) that S_0 is a solution of the OBIP when $P > 0$.

Additional analysis based in large part on (8) then serves to show that the linear fractional transformation of $\mathscr{S}^{p\times q}(\mathbb{D})$:

$$\{(\Theta_{11}S + \Theta_{12})(\Theta_{21}S + \Theta_{22})^{-1} : S \in \mathscr{S}^{p\times q}(\mathbb{D})\}, \tag{10}$$

is a complete description of the set of all solutions to the OBIP problem for $P > 0$.

A solution when $P \geq 0$ and singular may be obtained by a limiting argument; for a description of the set of all solutions in some typical singular cases, see, for example, [Br; BH; D1, Chapter 7].

The well-known problems of NP, CF (Carathéodory-Fejér), mixtures of the two, and others fit into this framework. In particular, the data of the NP problem is of the form (6) with

$$C = \begin{bmatrix} \xi_1 & \cdots & \xi_n \\ \eta_1 & \cdots & \eta_n \end{bmatrix} \quad \text{and} \quad A = \operatorname{diag}\{\omega_1^*, \ldots, \omega_n^*\}.$$

Therefore, by (4) and the preceding discussion it is solvable if and only if the $n \times n$ matrix P with ij entry

$$P_{ij} = \left\langle \frac{\xi_j}{\rho_{\omega_j}}, \frac{\xi_i}{\rho_{\omega_i}} \right\rangle - \left\langle \frac{\eta_j}{\rho_{\omega_j}}, \frac{\eta_i}{\rho_{\omega_i}} \right\rangle = \frac{\xi_i^* \xi_j - \eta_i^* \eta_j}{\rho_{\omega_j}(\omega_i)}$$

is positive semidefinite. This is the well-known Pick condition. Moreover, for this choice of data, formula (7) reduces to

$$[\xi_j^* \quad -\eta_j^*]\Theta(\omega_j) = 0, \tag{11}$$

thanks to (2), which is perhaps a little more transparent to track through, especially the first time.

6. Moral

The verification that $P > 0$ is a sufficient condition for the existence of a solution to the OBIP with suitably restricted data amounts to finding a J inner matrix Θ of McMillan degree n such that (7) holds. This can be done in many ways even though there is only one such Θ up to a J unitary constant $m \times m$ matrix factor on the right:

$$\Theta(\lambda) = I_m - \rho_\mu(\lambda) F(\lambda) P^{-1} F(\mu)^* J \tag{12}$$

for any choice of $\mu \in \mathbb{T}$. My own (and hence, of course, favorite) approach is based on reproducing kernel Hilbert spaces of a special kind which were introduced by deBranges [dB] (and adapted to the disk by Ball [Ba1]) for other purposes; see [D1, D2, D4]. Such a J inner Θ can, as we have already noted, be found in many other ways: Ball and Helton [Ba2, He2] invoke extensions of the Beurling-Lax-Halmos theorem to spaces with an indefinite inner product; Potapov [KPot] first characterized all solutions in terms of a fundamental matrix inequality; Krein and Langer [KL1, KL2] use the theory of operator extensions; Ball, Gohberg, and Rodman [BGR] make extensive use of realization theory for rational matrix functions and other state space methods. To ease the comparison with the latter, we remark that if F is of the special form (6) and $C^* = [C_{11}^* \; C_{21}^*]$ with $C_{11} \in \mathbb{C}^{p \times n}$ and $C_{21} \in \mathbb{C}^{q \times n}$, then S is a solution of the OBIP if and only if

$$\sum \text{residues } \{(\lambda I_n - A^*)^{-1} C_{11}^* S\} = C_{21}^*,$$

where the sum is taken over all the poles of $(\lambda I_n - A^*)^{-1}$ in $\mathbb{D}$.

The function $\Theta(\lambda)$ can also be built recursively out of "elementary J inner factors" of the basic form

$$\Theta_j(\lambda) = I_m + \{b_{\omega_j}(\lambda) - 1\} u_j (u_j^* J u_j)^{-1} u_j^* J, \qquad j = 1, \dots, n, \tag{13}$$

wherein

$$b_{\omega_j}(\lambda) = (\lambda - \omega_j)/(1 - \lambda \omega_j^*), \qquad u_j \in \mathbb{C}^n,$$

and $u_j^* J u_j > 0$. Thus for the NP problem, the ω_j are chosen to match the interpolation points and the u_j are chosen according to the rule $v_1 = u_1$ and

$$v_j^* J \Theta_1(\omega_j) \cdots \Theta_{j-1}(\omega_j) = u_j^* J \quad \text{for } j = 2, \dots, n,$$

where $v_j^* = [\xi_j^* \; \eta_j^*]$. This ensures that

$$v_j^* J \Theta_1(\omega_j) \cdots \Theta_j(\omega_j) = 0, \qquad j = 1, \dots, n.$$

Consequently,

$$\Theta = \Theta_1 \cdots \Theta_n \tag{14}$$

is a J inner matrix function of McMillan degree n which meets (11). The positive definiteness of P guarantees that $u_j^* J u_j > 0$ for $j = 1, \ldots, n$. (More details may be found, for example, in the proof of Theorem 4.2 of [D1] which gives a recursive construction of Θ which is applicable to the OBIP and on pp. 58, 65 for the NP problem and additional references.)

For a wider scope of applications and/or extensions of some of the approaches mentioned above, see [ABDS2, AD1, AD2, D3, D5] for more on reproducing kernel methods, [AB, KKY] for more on Potapov's method, [ABDS1, ABDS2] for more on the method of Krein and Langer, and Young [Y] for a very readable account of the NP problem in the spirit of Sarason.

Other methods for solving assorted classes of interpolation problems include: the momentum theorem of Rosenblum and Rovnyak [RR] which elaborates on an idea of Nudelman [Nu]; the band method of Gohberg and the reviewer; its extensions by Gohberg, Kaashoek and Woedermann, and most recently by Ellis, Gohberg, and Lay [EGL] (the latter contains references to the former); the use of the Nehari problem as in Arov and Krein [AK, Arov1, Arov2]; and the CLT (commutant lifting theorem), which, as we have already remarked, is the focal point of the volume under review. Before coming to that, however, a few more words on the Nehari problem are in order.

7. On solving the Nehari problem

Let Γ denote the operator which sends $\{\eta_0, \eta_1, \ldots\} \in l_2^q$ into $\{\xi_1, \xi_2, \ldots\} \in l_2^p$ by the rule

$$\xi_k = \sum_{j=0}^{\infty} \gamma_{k+j}\eta_j ,$$

for $k = 1, 2, \ldots$. Then the Nehari problem is solvable if and only if

$$\|\Gamma\| \le 1 . \tag{15}$$

The result is due to Nehari for the scalar case and Page [Pa] for the matrix and operator cases. A linear fractional representation of the set of all solutions when $\|\Gamma\| < 1$, and much more, is due to Adamjan, Arov, and Krein [AAK2, AAK3].

The proof of necessity is easy: If G is a solution, then the operator $\widehat{\Gamma}$ which maps $f = \sum_{j=0}^{\infty} \eta_j \lambda^j$ from $H_2^q(\mathbb{D})$ into $\{H_2^p(\mathbb{D})\}^\perp$ by the rule

$$\widehat{\Gamma} f = \underset{=}{q}' G f \tag{16}$$

is clearly a contraction. Therefore,

$$\|\Gamma\| = \|\widehat{\Gamma}\| \le 1 . \tag{17}$$

The proof of the sufficiency of this condition is harder. However, if the data $\gamma_1, \gamma_2, \ldots$ is square summable (which is, in fact, a necessary condition for the existence of a solution) and is subject to the restriction that $\sum_{j=1}^{\infty} \gamma_j \lambda^{-j}$ is a rational $p \times q$ matrix-valued function of McMillan degree n, then it can be expressed in the form

$$\sum_{j=1}^{\infty} \gamma_j \lambda^{-j} = D(\lambda)^{-1} E(\lambda),$$

where D is the product of n elementary Blaschke-Potapov $p \times p$ matrix inner factors (just like (13) but with $J = I$) and $E \in H_\infty^{p\times q}(\mathbb{D})$. Therefore, if there exists an $A \in H_\infty^{p\times q}(\mathbb{D})$ such that

$$\|D^{-1}E + A\|_\infty \leq 1,$$

then

$$S = E + DA \in \mathscr{S}^{p\times q}(\mathbb{D})$$

and

$$\underline{\underline{p}} S^* g = \underline{\underline{p}} E^* g \quad \text{for every } g \in H_2^p \ominus DH_2^p. \tag{18}$$

Conversely, if $S \in \mathscr{S}^{p\times q}(\mathbb{D})$ satisfies (18), then it is readily checked that

$$A = D^{-1}(S - E)$$

belongs to $H_\infty^{p\times q}(\mathbb{D})$ and hence that

$$\|D^{-1}E + A\|_\infty = \|D^{-1}S\|_\infty = \|S\|_\infty \leq 1.$$

In other words, if $g_1, \ldots, g_n$ is a basis for $H_2^p \ominus DH_2^p$ and

$$h_j = \underline{\underline{p}} E^* g_j \quad \text{for } j = 1, \ldots, n,$$

then the Nehari problem for such a special choice of the data is solvable if and only if there exists an $S \in \mathscr{S}^{p\times q}(\mathbb{D})$ such that

$$\underline{\underline{p}} S^* g_j = h_j \quad \text{for } j = 1, \ldots, n,$$

i.e., if and only if the $n \times n$ matrix P specified by (4), but for this choice of $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$, is positive semidefinite.

But now by (16),

$$\widehat{\Gamma} = \underline{\underline{q}}' D^{-1} E|_{H_2^q} \quad \text{and} \quad \widehat{\Gamma}^* = \underline{\underline{p}} E^* D|_{(H_2^p)^\perp}.$$

Therefore, since

$$(H_2^p)^\perp = D^*(H_2^p \ominus DH_2^p) \oplus D^*(H_2^p)^\perp,$$

it is readily checked that

$$\begin{aligned} \|\widehat{\Gamma}^*\|^2 &= \sup_{x_1, \ldots, x_n} \left\{ \left\| \underline{\underline{p}} E^* D D^* \sum_{j=1}^n x_j g_j \right\|^2 : \left\| \sum_{j=1}^n x_j g_j \right\|^2 = 1 \right\} \\ &= \sup_{x_1, \ldots, x_n} \left\{ \left\| \sum_{j=1}^n x_j h_j \right\|^2 : \left\| \sum_{j=1}^n x_j g_j \right\|^2 = 1 \right\} \end{aligned}$$

and hence that

$$\|\Gamma\| \leq 1 \quad \text{if and only if} \quad P \geq 0.$$

Thus at least for such special choices of the data $\gamma_1, \gamma_2, \ldots$, the advertised condition for the solvability of the Nehari problem emerges from the solvability of the OBIP discussed earlier. The general case requires limiting arguments or other methods. Adamjan, Arov, and Krein gave two proofs, one based on the theory of one-step extensions [AAK2, AAK3], and one based on the abstract theory of operator extensions for the scalar case [AAK1]. The latter was generalized to operator-valued functions by Adamyan [Ad]. Page [Pa] used the commutant lifting theorem. Another proof emerges from the solution of a more general completion problem by Katsnelson [K2], which in turn was motivated by the earlier work of Arocena, Cotlar, and Sadosky on the extension of "generalized Toeplitz" kernels.

8. Commutant Lifting

The Commutant Lifting Theorem. *Let $\mathscr{H}_i$ and $\mathscr{K}_i$, $i = 1, 2$, be two pairs of complex separable Hilbert spaces such that $\mathscr{H}_i \subset \mathscr{K}_i$ isometrically; let $T_i \in \mathscr{B}(\mathscr{H}_i)$; let $W_i \in \mathscr{B}(\mathscr{K}_i)$ be dilations of T_i:*

$$T_i^n = P_{\mathscr{H}_i} W_i^n|_{\mathscr{H}_i}, \qquad n = 0, 1, \ldots;$$

and let $X \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ intertwine T_1 and T_2: $XT_1 = T_2X$.

(A) *If the W_i are coisometries on $\mathscr{K}_i$, then there exists a $Y \in \mathscr{B}(\mathscr{K}_1, \mathscr{K}_2)$ such that:*

(A1) $YW_1 = W_2Y$,
(A2) $Y\mathscr{H}_1 \subset \mathscr{H}_2$,
(A3) $Yh_1 = Xh_1$ *for every* $h_1 \in \mathscr{H}_1$,
(A4) $\|Y\| = \|X\|$.

(B) *If the W_i are isometries on $\mathscr{K}_i$, then there exists a $Y \in \mathscr{B}(\mathscr{K}_1, \mathscr{K}_2)$ such that:*

(B1) $YW_1 = W_2Y$,
(B2) $Y(\mathscr{K}_1 \ominus \mathscr{H}_1) \subset \mathscr{K}_2 \ominus \mathscr{H}_2$,
(B3) $X = P_{\mathscr{H}_2} Y|_{\mathscr{H}_1}$,
(B4) $\|X\| = \|Y\|$.

Since W_i is a dilation of T_i if and only if W_i^* is a dilation of T_i^*, (B) can be obtained from (A) (and vice versa) by playing with adjoints.

Four proofs of the CLT are given in the book under review.

The lifting theorem can also be described in terms of extensions of a class of kernels of mixed Toeplitz-Hankel form. This circle of ideas has been extensively studied and applied by the Cotlar school. For a sample of their work see, for example, Cotlar and Sadosky [CS] and Arocena [Aroc1, Aroc2]. A more extensive list is provided in the book under review.

The usefulness of the CLT rests in large part on the fact that in many applications it is possible to identify Y in terms of a multiplication operator. For the OBIP this is achieved by the following fact.

Theorem. *Let T be a bounded linear operator from $H_2^p(\mathbb{D})$ into $H_2^q(\mathbb{D})$ such that $R_0T = TR_0$. Then there exists a $p \times q$ matrix-valued Φ which is analytic in $\mathbb{D}$ such that*

$$Tf = \underline{\underline{p}}\Phi^* f, \qquad \|T\| = \sup_{\omega \in \mathbb{D}} \|\Phi(\omega)\|_\infty .$$

9. Application of the CLT to the OBIP

In terms of the data of the OBIP, let

$$\mathscr{H}_1 = \text{span}\{g_j\colon j = 1, \dots, n\}, \qquad \mathscr{K}_1 = H_2^p(\mathbb{D}),$$
$$\mathscr{H}_2 = \text{span}\{h_j\colon j = 1, \dots, n\}, \qquad \mathscr{K}_2 = H_2^q(\mathbb{D}),$$

$T_i = R_0|_{\mathscr{H}_i}$, $W_i = R_0|_{\mathscr{K}_i}$, and X denote the linear operator which is defined by the rule

$$Xg_j = h_j, \qquad j = 1, \dots, n.$$

The operator X is well defined because of the presumed linear independence of the functions g_j, $j = 1, \dots, n$. Moreover, it is readily checked that $T_i\mathscr{H}_i \subset \mathscr{H}_i$, $W_i\mathscr{K}_i \subset \mathscr{K}_i$, $XT_1 = T_2X$, and that if P denotes the Pick matrix defined in (4), then

$$P \geq 0 \quad \text{if and only if} \quad \|X\| \leq 1.$$

Therefore, since W_i is a coisometric dilation of T_i, (A) of the CLT guarantees the existence of a bounded linear operator Y from $H_2^p(\mathbb{D})$ into $H_2^q(\mathbb{D})$ such that (A1)–(A4) hold. In particular, if $P \geq 0$, then $\|Y\| \leq 1$, and the last theorem further guarantees the existence of an $S \in \mathscr{S}^{p\times q}(\mathbb{D})$ such that

$$h_j = Xg_j = Yg_j = \underline{\underline{p}}S^* g_j$$

for $j = 1, \dots, n$.

One advantage of the CLT approach is that it guarantees the existence of a solution even if P is positive semidefinite and singular, without having to pass to limits. On the other hand, a description of the set of all solutions for $P > 0$, which is obtained through the parametrization of the set of all contractive intertwining dilations Y of the basic operator X, is a formidable calculation. For a relatively painless treatment which is based on a (not so quick) formula of Arov and Grossman [AG], see Arocena [Aroc1].

10. Application of the CLT to the Nehari problem

The crux of the matter is to establish the sufficiency of condition (15). To this end, it is convenient to first introduce the multiplication operator

$$M_\lambda\colon \sum f_j\lambda^j \to \lambda \sum f_j\lambda^j.$$

Then, in the CLT, let

$$\begin{aligned} \mathscr{H}_1 &= H_2^q(\mathbb{T}), & \mathscr{K}_1 &= L_2^q(\mathbb{T}), \\ \mathscr{H}_2 &= H_2^p(\mathbb{T})^\perp, & \mathscr{K}_2 &= L_2^p(\mathbb{T}), \\ T_1 &= M_\lambda|_{\mathscr{H}_1}, & W_1 &= M_\lambda \quad \text{on } \mathscr{K}_1, \\ T_2 &= \underline{\underline{q}}' M_\lambda|_{\mathscr{H}_2}, & W_2 &= M_\lambda \quad \text{on } \mathscr{K}_2, \end{aligned}$$

and define X by the rule

$$X\colon \sum_{j=0}^\infty \eta_j\lambda^j \to \sum_{k=1}^\infty \left(\sum_{j=0}^\infty \gamma_{k+j}\eta_j \right) \lambda^{-k}.$$

It is now readily checked that if $\|\Gamma\| \leq 1$, then $\|X\| \leq 1$ and $XT_1 = T_2X$. Therefore, since W_j is an isometric dilation of T_j for $j = 1, 2$, formulation

(B) of the CLT guarantees the existence of a $Y \in \mathscr{B}(\mathscr{K}_1, \mathscr{K}_2)$ with $\|Y\| = \|X\| \leq 1$ such that $YW_1 = W_2Y$. Thus by a variant of the last theorem in the preceding section (see [FF, Theorem 1.1]), there exists a $G \in L_\infty^{p\times q}(\mathbb{T})$ with $\|G\|_\infty = \|Y\|$ such that $Yf = Gf$ for every $f \in \mathscr{K}_1$. Item (B3) then guarantees that

$$Xf = \underset{=}{q}'Gf \quad \text{for every } f \in \mathscr{K}_1 .$$

But this is the same as to say that

$$\sum_{k=1}^{\infty}\left(\sum_{j=0}^{\infty}\gamma_{k+j}\eta_j\right)\lambda^{-k} = \sum_{k=1}^{\infty}\left(\sum_{j=0}^{\infty}G_{-k-j}\eta_j\right)\lambda^{-k},$$

from which in turn it is readily seen that $\gamma_j = G_{-j}$ for $j \geq 1$ by matching Fourier coefficients. This completes the proof of the stated version of the Nehari theorem by the CLT.

11. Networks, systems and control

In the last decade or so, the interpolation problems sketched above (as well as more general formulations) have been extensively applied to problems in the mathematical theory of networks, systems, and control. This interaction has in turn spurred new investigations and developments. Some indications of these connections are given in Chapter 9 of the book under review; see also Chapters 23–25 of [BGR] and [He2].

The engineering literature itself divides roughly into two categories: applications of NP-type interpolation and applications of AAK, even though, as noted earlier, these two sets of problems are not entirely independent. For samples of the former see Youla and Saito [YS]; Helton [He1]; Delsarte, Genin, and Kamp [DGK]; Dewilde and Dym [DD]; Tannenbaum [Ta]; Francis and Zames [FrZ]; Chang and Pearson [CP]; Georgiou and Khargonekar [GeK]; Tadmor [Tad1]; Limebeer and Anderson [LA]; Kimura [Ki1–Ki3]; Limebeer and Green [LGr, GrL]; and Limebeer and Kasenally [LK].

Applications of AAK to electrical engineering problems rest largely on:

(1) the fundamental formula

$$\inf\{\|G - F\|_\infty : F \in H_{\infty,k}^{p\times q}\} = \sigma_k , \tag{19}$$

where the σ_k, $k = 0, 1, \ldots$ are the singular values of a Hankel operator based on the given $G \in L_\infty^{p\times q}$ and where $H_{\infty,k}^{p\times q}$ differs from $H_\infty^{p\times q}$ by "at most k poles", and

(2) linear fractional representations of the set of all F which either attain or "come close to" attaining the stated infimum.

Formula (19) was established in [AAK2] for the scalar case ($p = q = 1$) and general k and for general p and q but $k = 0$ in [AAK3]. Engineering applications (and extensions of these results to $k > 0$ for the nonscalar case) typically focus on the case where F and G are rational. Moreover, more often than not, engineers prefer to work with half-plane versions of these problems. Some early papers in this direction are those of Genin and Kung [GK], Bultheel and Dewilde [BD], and Kung and Lin [KL], though Helton used some of the AAK results some years before in this study of broadband matching [He1]. However, the most influential work of this period was undoubtedly Glover's 1984 paper in which he established (19) for rational matrix-valued F and G in an elementary self-contained way in the language of state space and, hence, made these results accessible to a much wider cross-section of the engineering community; for an updated version of this paper see [Glo]. For additional information, the lecture notes of Francis [Fr] and the expository article by Francis and Doyle [FrD] (both of which seem to have been influenced by Doyle's 1984 Honeywell Workshop notes) are highly recommended. These last three sources give a pretty good picture of the role of AAK in H_∞ control (and for $k > 0$ in (19), model reduction). Good supplementary sources are the collected lecture notes of the special session on H_∞-Control which was held in Como, Italy, in June 1990 [FFHKP] and the papers of Ball and Ran [BR], Ball and Cohen [BaC], and Curtain and Zwart [CZ].

Recent years have witnessed a shift in emphasis in the H_∞ control literature towards other approaches which lead to simpler expositions of the basic theory and formulae and which are also easier to implement numerically (because of lower-dimensional state space formulae); see Glover and Doyle [GloD], Green [Gr], and the "multinationals" [DGKF, GGLD, GLDKS] for the state of the art (or, if you wish, the art of the state).

To round out the picture from yet other points of view, the papers by Foias and Tannenbaum [FT], Georgiou and Smith [GeS], and Tadmor [Tad2] are suggested.

12. The poem is everything. Publication is nothing

These sentiments, which were expressed by Emily Dickinson more than a century ago, seem to have gone out of fashion. Therefore, perhaps a few words of explanation on the bibliography of this review are in order. I have not striven for completeness but rather have attempted to indicate some of the main developments in the last twenty years or so by drawing representative samples from the literature. Thus, for example, I have not cited the many joint papers of Ball and Helton (or most of the joint work of Ball and Ran in which the Ball-Helton machinery was applied to rational matrix-valued functions) but have chosen instead two representative expository works: [Ba2] and [He2]. Also, for sequences of papers, I have more often than not listed the latest one only.

The classical works from the turn of the century through Nehari have not been referenced on the grounds that they are both well known and are frequently referenced elsewhere. Most of the books cited here (including the book under review) have bibliographical notes which are well worth reading. In addition, the collection [HSP] of classic papers contains a very fine survey of the mathematical literature on interpolation theory by the "two Bernds", Fritzsche and Kirstein. (The papers and the survey are in German; an English translation of Schur's famous paper is available in volume 18 of the OT series.)

The cited literature on engineering applications is also woefully incomplete. Here too I have tried to draw on representative (and where possible, expository) samples rather than a complete list. The books in preparation by Zhou, Doyle and Glover, Green and Limebeer, and Kimura will undoubtedly help to fill this gap. Two forthcoming volumes in the OT series, one dedicated to the seventieth birthday of M. S. Livšic and one to the memory of V. P. Potapov, will also serve to clarify further the important role of these two outstanding mathematicians and their students to the development of the mathematical theory of networks and systems.

13. The book

This book is undoubtedly a labor of love. It is a painstakingly carefully written exposition with meticulous attention to detail and a responsible concern for the reader as evidenced by numerous parenthetic explanatory remarks, a good index, a directory of symbols, and a road map of how to get through various topics with a minimum of pain. Moreover, it seems to be remarkably free of errors.

True to its title, the major theme of the book is the commutant lifting theorem and its application to an assorted collection of interpolation problems. But it is much more than that. It incorporates a full development of the commutant lifting theorem, starting from scratch, and numerous ancillary facts in the general setting of operator-valued function theory. The latter is particularly welcome since the now classic source book [SNF3] of Sz.-Nagy and Foias has long been out of print. In addition to the interpolation problems touched upon here, the book discusses the interpolation problem of Loewner, numerous algorithms, applications to H^∞ optimization and much more, and more often than not, operator-valued interpolation (as opposed to matrix or vector). It does not treat bitangential interpolation problems, extension problems in spaces with an indefinite metric, or continuous completion problems akin to Krein's problem for extending a function defined on an interval subject to certain positivity constraints.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 1, July 1994

0273-0979/94 $1.00 + $.25 per page

Smoothness and renormings in Banach spaces, by Robert DeVille, Gilles Godefroy and Vaclav Zizler. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993, xii + 220 pp., $130.00. ISBN 0-582-07250-6

Renorming theorems has been the key to applying Banach space ideas to other areas of mathematics almost from the beginning of the theory. In the work of Clarkson, Dunford, and Morse we find simple renormings clearing the way to differentiability results. Krein, Krasnoselskii, and Milman used renormings to derive basic properties about openings of subspaces in their work on defect numbers of operators, results rediscovered and given substantially more complicated proofs (unaided by renormings) by Tikhomirov in his study of widths of sets and best approximants. Kadets renormed separable spaces in a locally uniformly convex manner and, using this, introduced good coordinate systems in separable spaces on his way to proving all separable infinite-dimensional Banach spaces are homeomorphic. Bonic and Frampton showed c_0 has a C^∞ smooth norm (away from 0) and used this, along with companion results for other classical spaces, to carry out their work in Banach manifolds. Pisier tied vector-valued versions of the Burkholder-Davis-Gundy inequalities for martingale difference sequences to the vector space's superreflexivity; probability gained as did Banach space theory.