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Fundamentals of dynamical systems and bifurcation theory, by Milan Medved, translated from Slovak by J. Hajnovicova and D. Halasova. Adam Hilger, Bristol, Philadelphia, and New York, 1992, viii + 293 pp., $66.00. ISBN 0-7503-0150-3

Dynamical systems are currently one of the most popular scientific areas. This is particularly true for the area of nonlinear dynamics, which very often is more or less identified with chaos theory. Not only is the name chaos not justified for any nonlinear phenomenon such as elementary bifurcations, but the popularity of the subject also led to an avalanche of fuzzy publications with eye-catching illustrations and speculative comments. Here, though, is a book which is completely different: a rigorous mathematical treatment of the fundamentals of differentiable dynamical systems and bifurcations, a systematic introduction of basic terminology and mathematical theory presented in a general and rather abstract framework with relatively few schematic figures included.

The theory of dynamical systems is closely related to the geometric or qualitative theory of differential equations which dates back to Poincaré [13]. This theory studies the solutions from the point of view of the geometry of the corresponding set of trajectories or orbits in phase space rather than of quantitative features of single solutions. In particular, it provides a means to verify the reliability of quantitative computations and computer simulations. An autonomous first-order system of ordinary differential equations generates a continuous dynamical system on the corresponding state space, given by the flow map. The flow map depends on the independent variable, say, time t, and assigns to each initial state the solution at time t. Furthermore, a time-periodic first-order system of ordinary differential equations is completely described by the discrete dynamical system which is generated by the period-map on the state space. Iterating the period-map gives the solutions at equally spaced instants of time which differ by one period of the system. Analogously, a continuous system near a periodic orbit can be described by a discrete system. Here the generating diffeomorphism is the first return map or Poincaré map corresponding to some transversal section of the periodic orbit.

A key feature of the mathematical theory of dynamical systems is to ensure some kind of robustness or structural stability of qualitative properties of the trajectories when the system is perturbed slightly. This is important, because in applications, model equations always involve approximations of certain quantities. Of course, such a concept depends on the kind of perturbations which are taken into account and requires some notion of equivalence of orbit structures, either locally or globally. Usually, one considers spaces of C^k-vector fields and C^k-diffeomorphisms equipped with a Whitney type C^k-topology, $k \geq 1$. An appropriate equivalence relation is orbital topological equivalence. According to this, two systems are equivalent in a region of their phase space if there exists a homeomorphism which maps orbits of one onto orbits of the other one and preserves the direction of the orbits. Having such a notion of equivalence, one is also interested in a classification of the corresponding equivalence classes according to the geometry of the orbit structure. In fact, for structurally stable C^k-systems on smooth, orientable, compact manifolds of dimension two, the equivalence classes can be completely classified. Moreover, these systems are open and dense in the space of all C^k-systems; i.e., structural stability is a generic property here. These results were proven by M. M. Peixoto [12]; however, on higher dimensional manifolds the structurally stable systems are not dense (cf. Smale [14]). So in general it does not make sense to restrict the study and classification of the dynamics just to structurally stable systems. Rather, it is natural to consider so-called generic properties. A property is said to be *generic* if and only if it is satisfied by all systems of a given class which belong to a residual or, as called in the present book, massive subset. Note that spaces of C^k-vector fields and C^k-diffeomorphisms are Baire spaces with respect to the Whitney topology. Hence, countable intersections of open and dense subsets are still dense. Subsets given by this kind of an intersection are called residual. A famous example is provided by the Kupka-Smale systems on smooth manifolds (Smale [15]). It is to be noted, though, that even from the generic point of view a geometric classification of the global behaviour of dynamical trajectories in a dimension greater than or equal to three seems to be hopeless in view of the complex structures which are possible (chaotic behaviour).

On the other hand, locally, structural stability is generic, and one has a complete generic classification of both continuous and discrete finite-dimensional systems based on the flow box theorem and the Hartman-Grobman theorem. The former says that away from any equilibrium or periodic point trajectories are parallel in the topological sense. The latter says that near a hyperbolic equilibrium or periodic point a dynamical system is orbitally topologically equivalent to a simple linear system, called the normal form. As a matter of fact, there is even a generalization of this local normal form to the nonhyperbolic, i.e., nongeneric case (cf. Palmer [11], Sositaisvili [16]) based on the center-manifold theorem. This generalization is also called the principle of reduction onto the centre manifold, since the resulting normal form is linear outside the center manifold. In addition, normal form techniques of Birkhoff [2], Elphick et al. [4], Takens [17], and others provide a means to achieve the most convenient form of the nonlinear part in order to analyze the local structure of the trajectories. By means of Poincaré maps, the local theory also carries over to neighborhoods of periodic orbits in case of continuous systems.

Let us now consider parameterized families of dynamical systems. These are the objects which are studied in bifurcation theory. Bifurcations are changes of the topological structure of trajectories when the parameter is changed. Bifurcation points are points in the parameter space at which such changes occur. Of course, at such points the system cannot be structurally stable. In fact, properties which are nongeneric for single systems can occur generically for certain parameter values in families of systems. However, just as in the case of single systems, in general, global structural stability with respect to any reasonable topology and equivalence relation is not a generic property here either. As far as a local generic theory of bifurcations is concerned, R. Thom's "catastrophe theory" [18] is a cornerstone. He classified singularities, i.e., critical points of smooth potential functions depending on parameters (or rather germs of such functions) and constructed normal forms (universal unfoldings). This amounts to studying bifurcation of equilibrium points of gradient systems. Later, Golubitsky and Schaeffer [6, 7] extended this theory on the basis of a different equivalence relation including a distinguished parameter—namely, contact equivalence—to more general systems and, in particular, to systems with symmetries. Also, they used the Liapunov-Schmidt method to include the consideration of periodic orbits (Hopf bifurcation). In the present book, the theory of local bifurcations near equilibrium points or periodic points is based on generic properties and normal forms for the linearized system, i.e., for parameter-dependent matrices (cf. Arnold [1]). Moreover, nondegeneracy for the nonlinear terms is assumed in order to prove normal form, bifurcation, and dynamical stability results. This point of view is rather common in the literature (cf. Chow and Hale [3], Guckenheimer and Holmes [8]). The hope is to obtain a generic classification of the local bifurcations in this way. However, up to now a complete generic classification has been worked out only for one-parameter families of systems. In this case, only bifurcation from simple real or complex conjugate eigenvalues is generic. For multiparameter systems just partial results are known. One of the most far-reaching ones is the Takens-Bogdanov theorem for two-parameter systems.

The book under review is concerned with the generic theory of dynamics and bifurcations as outlined above. The frame in which this theory is presented is general differentiable dynamical systems on finite-dimensional smooth manifolds. Both discrete and continuous systems are considered simultaneously, with a few exceptions such as the center manifold theory. The latter is developed only for a particular class of local vector fields.

The author really starts from scratch. The first chapter is a compendium of basic notation and material from algebra, topology, mathematical analysis, and differential equations, mostly without proofs but with abundant references. Next follows a slightly more discursive account of the basics of differential topology. It was through the generic theory that ideas and concepts from differential topology were introduced into the subject of dynamical systems and bifurcations. Credit for this should be given to Smale and Thom, among others. In particular, the important concept of "transversality" is discussed in the book. Also, a valuable section on stratification of algebraic and semialgebraic varieties is included. Stratification theory is not easily available elsewhere in dynamical systems literature. It is needed in a section on generic properties of parameter-dependent matrices, which is presented in great detail. Notice that stratification theory has also been used recently in connection with the method of orbit space reduction in equivariant bifurcation theory (see Field [5], Menck [10]). This is not an issue in the book, since equivariant bifurcation problems are not generic in the general sense.

The heart of this book is three chapters on vector fields and dynamical systems, invariant manifolds, and generic bifurcation of vector fields and diffeomorphisms, respectively. These contain a lot of very careful analysis. Proofs of various basic assertions of the local theory are worked out in full detail. For proofs of the more sophisticated parts such as C^k-smoothness of center manifolds, $k \geq 1$ (cf. also [9]), or the Neimark-Sacker theorem on bifurcation of closed invariant curves for diffeomorphisms, the reader is referred to the literature. Also, several historical remarks are added. Interestingly enough, references to the literature cover contributions from Eastern Europe, the former Soviet Union, as well as the West. The book ends with some notes on global and chaotic dynamics, including the Smale horseshoe, Silnikov bifurcation, and attractors of Lorenz type.

As already pointed out, the approach of the book is very formal. It requires familiarity with the language of set theory. The book contains an abundance of formal definitions followed by precise statements of the results, while there is a shortage of illustrations and instructive examples. Therefore, it seems to be unlikely that it will be attractive to researchers outside mathematics who need to solve practical problems of a dynamical character, i.e., to those to whom it claims to be directed, other than graduate students in mathematics. Also, it is hard to imagine that this is a text by which the field of dynamical systems could be approached unprepared. After all, the English translation from Slovak, while good, is not perfect. For example, some mishandling of the definite and indefinite article does lead to genuine ambiguity over uniqueness in places. In any case, it is good for the subject that there are authors who care about thorough and reliable mathematical foundations.

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Topological rings, by Seth Warner. North Holland Math Studies, vol. 178, North-Holland, Amsterdam, 1993, x + 498 pp., $142.75. ISBN 0-444-89446-2

This volume should be considered in conjunction with the author's preceding book [8]; they form a natural pair. There is even a formal connection: pages 487–488 of *Topological rings* consist of errata for *Topological fields*. *Topological fields* has not been reviewed by this *Bulletin*. For the reader's possible convenience, here are two reviews: 90i: 12012 in MR and vol. 683, 12014 in the Zentralblatt.

I salute the author for publishing a total of $563 + 498 = 1061$ pages of sound, scholarly exposition within a four-year period.

The thesis [1] of van Dantzig marked the birth of topological rings as a new discipline. From the start there was a virtually complete dichotomy between the connected and totally disconnected cases. If R is a topological ring, the connected component I of 0 forms a closed two-sided ideal, and thus we have three things to study: the connected ring I, the totally disconnected ring R/I, and the extension problem that arises.

The connected case is a vast field including, for instance, all Banach algebras. Indeed, the five examples (pp. 3–4) of topological rings presented at the beginning of the book are all Banach algebras. Nevertheless, Banach algebras have a very low profile in the book and do not even appear in the index. It is the totally disconnected case that dominates. The major motivating example is the ring of p-adic integers, along with its quotient field. The first of these is a compact ring, the second a locally compact field. Van Dantzig [1] initiated the study of locally compact division rings. The connected case was fully treated by Pontrjagin [4] and given high visibility in his well-known book [5]. Jacobson treated the totally disconnected case definitively in [2] and, in collaboration with Taussky [3], gave a big push to the general theory of locally compact rings by fully exploiting the structure of locally compact abelian groups. About thirty years later the subject reached a climax when Skornjakov [6] exhibited "wild" simple locally compact rings. An indication of the publication explosion in mathematics is that in Small's collection [7] there is a whole section (no. 29.01, pp. 847–854) on locally compact rings and modules, comprising thirty-eight papers, and that brings us up to only 1979.