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BOOK REVIEW

Topological rings, by Seth Warner. North Holland Math Studies, vol. 178, North-Holland, Amsterdam, 1993, x + 498 pp., \$142.75. ISBN 0-444-89446-2

This volume should be considered in conjunction with the author's preceding book [8]; they form a natural pair. There is even a formal connection: pages 487–488 of *Topological rings* consist of errata for *Topological fields*. *Topological fields* has not been reviewed by this *Bulletin*. For the reader's possible convenience, here are two reviews: 90i: 12012 in MR and vol. 683, 12014 in the Zentralblatt.

I salute the author for publishing a total of $563 + 498 = 1061$ pages of sound, scholarly exposition within a four-year period.

The thesis [1] of van Dantzig marked the birth of topological rings as a new discipline. From the start there was a virtually complete dichotomy between the connected and totally disconnected cases. If R is a topological ring, the connected component I of 0 forms a closed two-sided ideal, and thus we have three things to study: the connected ring I, the totally disconnected ring R/I, and the extension problem that arises.

The connected case is a vast field including, for instance, all Banach algebras. Indeed, the five examples (pp. 3–4) of topological rings presented at the beginning of the book are all Banach algebras. Nevertheless, Banach algebras have a very low profile in the book and do not even appear in the index. It is the totally disconnected case that dominates. The major motivating example is the ring of p-adic integers, along with its quotient field. The first of these is a compact ring, the second a locally compact field. Van Dantzig [1] initiated the study of locally compact division rings. The connected case was fully treated by Pontrjagin [4] and given high visibility in his well-known book [5]. Jacobson treated the totally disconnected case definitively in [2] and, in collaboration with Taussky [3], gave a big push to the general theory of locally compact rings by fully exploiting the structure of locally compact abelian groups. About thirty years later the subject reached a climax when Skornjakov [6] exhibited "wild" simple locally compact rings. An indication of the publication explosion in mathematics is that in Small's collection [7] there is a whole section (no. 29.01, pp. 847–854) on locally compact rings and modules, comprising thirty-eight papers, and that brings us up to only 1979.

There is another stream of work that can be regarded as flowing from the example of the ring of p-adic integers: local rings. Chapter 5 (pp. 166–205) can be recommended as a good introduction to this important subject. (By a coincidence, Chapter 5 of *Topological fields*, dealing with valuations, is another useful self-contained introduction; the Zentralblatt review described it as a book within a book.)

There are sixteen pages of historical notes; they follow up the historical notes in *Topological fields*. In the latter the author disclaimed competence as a historian. He was too modest. Combined, these historical notes will surely stand for a long time as definitive.

The body of the book is also admirable for tracing the threads of the (often clumsy) first proofs and then bringing to the reader the best current proofs.

The two books belong in every mathematical library and in the libraries of individuals who can afford them.

References

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