There is another stream of work that can be regarded as flowing from the example of the ring of p-adic integers: local rings. Chapter 5 (pp. 166–205) can be recommended as a good introduction to this important subject. (By a coincidence, Chapter 5 of *Topological fields*, dealing with valuations, is another useful self-contained introduction; the Zentralblatt review described it as a book within a book.)

There are sixteen pages of historical notes; they follow up the historical notes in *Topological fields*. In the latter the author disclaimed competence as a historian. He was too modest. Combined, these historical notes will surely stand for a long time as definitive.

The body of the book is also admirable for tracing the threads of the (often clumsy) first proofs and then bringing to the reader the best current proofs.

The two books belong in every mathematical library and in the libraries of individuals who can afford them.

References

1. D. van Dantzig, *Studien over topologische algebra*, H. J. Paris, Amsterdam, 1931. (Dutch)
2. N. Jacobson, *Totally disconnected locally compact rings*, Amer. J. Math. **58** (1936), 433–449.
3. N. Jacobson and O. Taussky, *Locally compact rings*, Proc. Nat. Acad. Sci. U.S.A. **21** (1935), 106–108.
4. L. S. Pontrjagin, *Über stetige algebraische Körper*, Ann. of Math. **33** (1932), 163–174.
5. ———, *Topological groups*, English transl., 2nd Russian ed., Gordon and Breach, New York, 1966.
6. L. A. Skornjakov, *Einfache lokal bikompakte Ringe*, Math. Z. **87** (1965), 241–251.
7. L. Small, *Reviews in ring theory* 1940–1979, Amer. Math. Soc., Providence, RI, 1981.
8. S. Warner, *Topological fields*, North-Holland, Amsterdam, 1989.

Irving Kaplansky
Mathematical Sciences Research Institute

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Subfactors and knots, by V. F. R. Jones. CBMS Regional Conference Series in Mathematics, vol. 80, American Mathematical Society, Providence, RI, ix + 113 pp., \$43.00. ISBN 0-8218-0729-3

I. In search of a knot polynomial

This book by Vaughan Jones should be on the bookshelf of anyone who is curious about one of the twentieth century's most wonderful contributions to mathematics.

The subject is startling, and it remains startling to its practitioners, even ten years since the discovery of the Jones polynomial. A little history may help to give the flavor of this matter from the point of view of a topologist.

I.

II.

III.

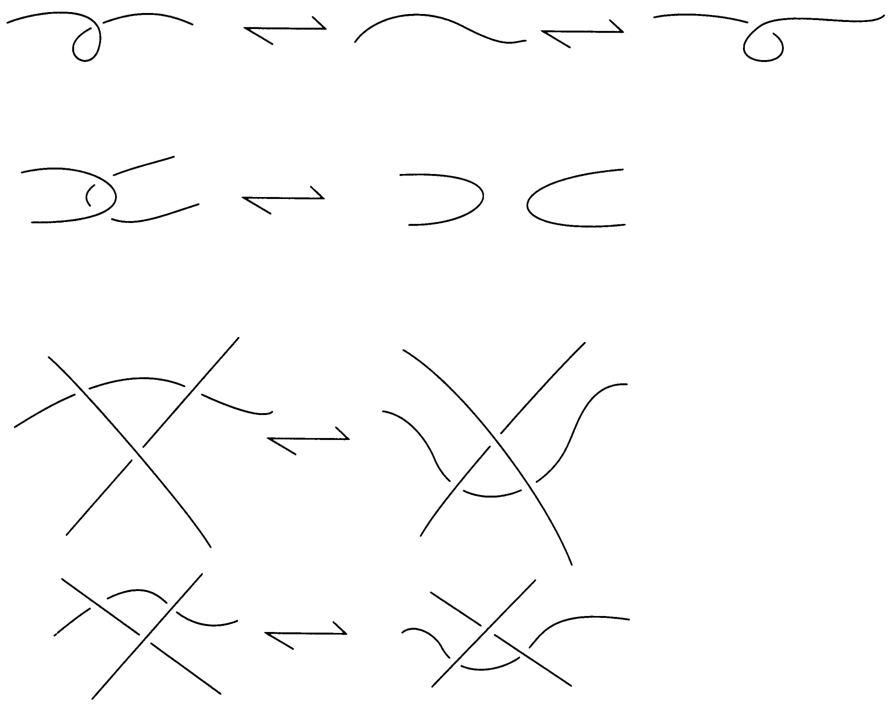


FIGURE 1

Alexander discovered a polynomial invariant of knots and links in the 1920s. He defined his invariant as the determinant of a certain matrix associated with the diagram of the knot or link [1]. He proved that up to a sign and powers of the polynomial variable t, his determinant was invariant under the Reidemeister moves. The Reidemeister moves, shown in Figure 1, are a set of transformations on diagrams for knots and links that encapsulate the concept of ambient isotopy. Two knots or links are ambient isotopic in three-dimensional space if and only if their diagrams (obtained by projection) can be transformed one into the other by a finite sequence of Reidemeister moves. The moves are performed locally in the diagrams, changing only the small part of the diagrams as shown in Figure 1. Reidemeister's Theorem [24] about these moves reduces a complex three-dimensional problem to a combinatorial question about the properties of a diagrammatic formal system—the link diagrams.

Alexander felt that the approach to knot theory via the Reidemeister moves was of sufficient importance to give it full precedence in his paper. A hint of the group-theoretic origins of the polynomial can be found at the end of Alexander's paper. There one sees that he had considered the cyclic covering spaces of the knot complement and their fundamental groups and that he did not quite articulate the concept of the infinite cyclic covering space where the algebra related to this polynomial lives most comfortably [22, 26].

Subsequent generations of mathematicians, living under the thrall of groups, homology, algebraic topology, and the lure of the higher dimensions, unraveled the group-theoretic nature of Alexander's polynomial. This culminated in the work of Fox [6], who, with his invention of the Free Differential Calculus, showed how to define the Alexander polynomial as an invariant of the fundamental group of the knot complement. Now one knew the true nature of this invariant. It was an algebraic characteristic of the fundamental group, and that group was a natural invariant of the complementary space of the knot. Earlier, Seifert had shown how to obtain Alexander's polynomial from an orientable spanning surface for the knot. This Seifert method was open to a geometric interpretation that led directly to the infinite cyclic covering of the knot complement and so, by that door, right back to the last pages of Alexander's original paper.

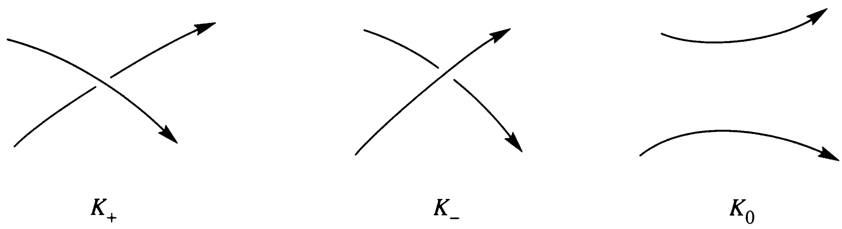
All of these developments meant that Alexander's polynomial was in the center of an exciting series of developments of geometric topology. The generality of the methods of definition led to corresponding definitions for Alexander polynomials and Seifert forms for higher-dimensional knots.

Much of this research received an initial impetus from the seminal work of Fox and Milnor [7] on knot concordance (the old term was cobordism), and the methods were integral to later developments in differential topology with the recognition and construction of exotic spheres by Milnor in the late 1950s. Through the work of Brieskorn (see [21]), exotic spheres appeared as links of complex algebraic singularities, and the Alexander polynomial played a key role in the structure of these knotted manifolds. In this stage a hint of physics appeared: The monodromy of the Brieskorn algebraic varieties had originally been calculated by Pham for the sake of the evaluation of certain Feynman integrals.

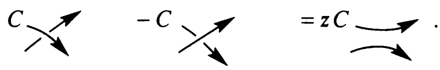
One would think that the Alexander polynomial was well understood, perhaps fully understood. What more could one do? Then John Horton Conway published a paper in 1970 [5]. His paper gave remarkably efficient ways to enumerate knots, a method of calculation of Alexander polynomials involving tangles (little boxes with four strings sticking out and some weaving pattern inside), fractions, continued fractions, and a method for calculating Alexander polynomials that did not involve any determinants or any mathematics at all—except playing about with the diagrams. In fact, a careful reader of Conway's paper would discover that the polynomial invariants in that paper were purported to be just a bit better than the Alexander invariants. Conway's polynomials were well defined, period. There was no ambiguity about sign or a power of a variable. This sign-normalization made it possible for Conway's invariants to detect the difference between many links and their mirror images. A link is an embedding of many circles into three-space. A knot has one component. Neither Conway's nor Alexander's polynomials could detect chirality (mirror-image difference) for knots. Conway gave the algorithms; he did not include their proofs. The paper was short and cryptic, and it was virtually ignored for almost ten years.

Conway's method for the one-variable Alexander polynomial can be axiomatized as follows: The Conway polynomial is denoted $C_K(z)$ where K is an oriented knot or link (orientation means that each strand is assigned a preferred direction) and z is the polynomial variable. $C_K(z)$ is a polynomial with integer coefficients, and it satisfies the following axioms.

0. $C_U(z) = 1$ when U is an unknotted circle.
1. If K and L are ambient isotopic links, then $C_K(z) = C_L(z)$.
2. (Exchange Identity) If K_+, K_-, K_0 are three links that differ at the site of one crossing as shown below, then $C_{K_+}(z) - C_{K_-}(z) = zC_{K_0}(z)$.



This last property is often abbreviated by the diagrammatic equation



Here the little diagrams denote the larger diagrams K_+, K_-, K_0 as indicated above.

It is by no means obvious that Conway's axioms are consistent or that they do model a version of the Alexander polynomial. In fact, a version of the exchange identity is proved in Alexander's original paper, but it is rendered difficult as an algorithm by the fact that the original Alexander polynomial is defined only up to a sign and up to multiplication by a power of its variable. With this as a clue, it was possible to reformulate Alexander's original determinant as a state summation (on combinatorial states of the link diagram) and thereby obtain one model for Conway's axioms [14]. (See also [13, 15].)

In 1984 Vaughan Jones discovered a new invariant of knots and links, based on a representation of the Artin Braid group to a von Neumann algebra [10, 11]. Jones proved that this polynomial, $V_K(t)$, satisfied an exchange identity of the following type:

$$t^{-1}V_{K_+}(t) - tV_{K_-}(t) = (\sqrt{t} - \sqrt{t}^{-1})V_{K_0}(t).$$

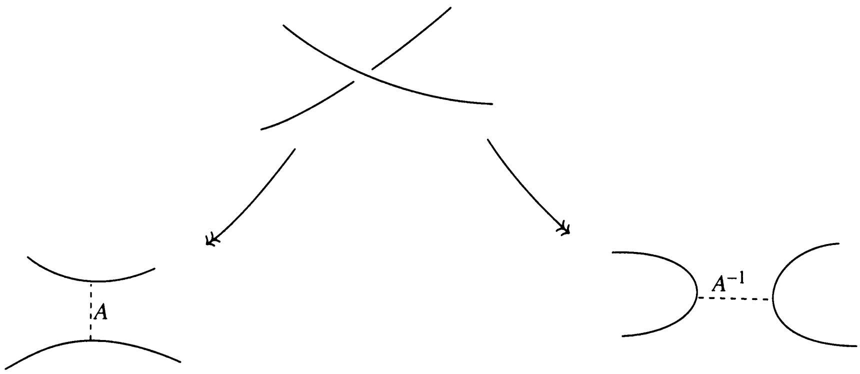
It then dawned on the community [8, 23, 19] that the Conway formulation had vast generalizations. A new era began in knot theory and in low-dimensional topology.

It turns out [17] that this original Jones polynomial has a state summation model similar (but simpler!) to the state summation already mentioned for the Conway-Alexander polynomial. This "bracket" model for the Jones polynomial is based on the axioms below:

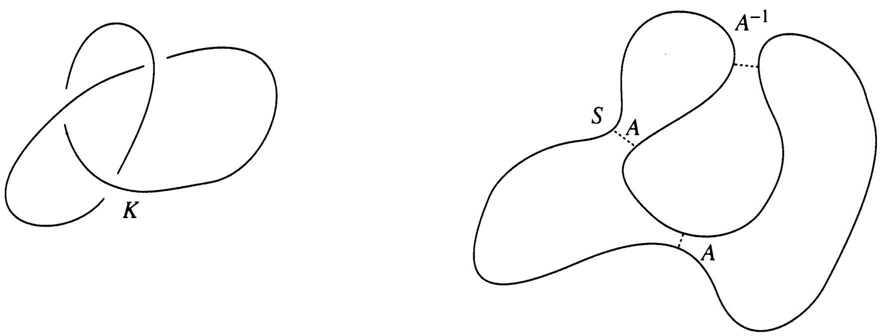
0. $\langle U \rangle = 1$ for an unknotted circle U.
1. $\langle \times \rangle = A\langle \asymp \rangle + B\langle \supset\subset \rangle$.
2. $\langle K\ O \rangle = d\langle K \rangle$ where $K\ \ O$ denotes a diagram with an extra circle in it disjoint from the rest of the diagram.

Calculations associated with these axioms are always well defined on diagrams but not invariant under the Reidemeister moves for arbitrary values of A, B, d. It is easy to see that if $B = A^{-1}$ and $d = -A^2 - A^{-2}$, then $\langle K \rangle$ is invariant under the second and third Reidemeister moves. A normalization of $f_K(A) = (-A^3)^{-w(K)}\langle K \rangle$ produces a Laurent polynomial $f_K(A)$ that is invariant under all three Reidemeister moves. (Here $w(K)$, the *writhe*, is the sum of the signs of the crossings in the link diagram.) It is then not hard to prove that $V_K(t) = f_K(t^{-1/4})$. Thus the bracket polynomial gives a state summation model for the original Jones polynomial.

The bracket model is a summation over states of the diagram in the following sense: A state S of an (unoriented) diagram K is a choice of smoothing of each of the crossings of K, as shown below.



Each smoothed crossing is labeled A or A^{-1} by the convention indicated above. Thus a state is a labeled collection of Jordan curves in the plane.



$\langle K|S\rangle = A^2A^{-1} = A, \qquad \|S\| = 1.$

We call the labels on the state its *vertex weights* and define $\langle K|S\rangle$ to be the product of the vertex weights of the state S. We let $\|S\|$ denote the number of Jordan curves in S. Then $\langle K\rangle$ is given by the state summation formula

$$\langle K\rangle = \sum_S \langle K|S\rangle\, d^{\|S\|-1}.$$

This state summation can be compared with the state summations (partition functions) in statistical mechanics. In the case at hand it shows [18] that the Jones polynomial is directly related to the formalism of the Potts model [3]. Here we are imagining that the link diagram is a little physical system with different modes of (say) magnetization on the regions of the diagram. In this case the loops in the state are the analogs of the boundaries of regions of constant polarity; that is, they are the boundaries of the magnetic domains. The von Neumann algebra that is instrumental in defining $V_K(t)$ makes its appearance naturally in the context of this state model and for the same reasons that it previously appeared in the context of the Potts model.

Jones's original approach to the new invariant was intimately tied with statistical mechanics. Once it became apparent that there were direct connections between statistical mechanics and the Jones polynomial, a host of relationships opened up. Many of these relationships and new invariants are described in Jones's book, so we shall not try to describe them here. This section of the review is designed to give historical context and to whet the reader's appetite for the book itself. In the next section, we give a brief description of its contents.

II. Subfactors and knots

This book is a set of lectures on von Neumann algebras, braid groups, and invariants of knots and links. Vaughan Jones discovered extraordinary connections among these subjects. It is fitting to have his exposition of these ideas. To this reviewer's knowledge, this book constitutes the only exposition other than [11] of the Jones polynomial for a general mathematical audience that starts with an introduction to von Neumann algebras.

The book begins with two lectures on von Neumann algebras and subfactors. By the end of the second lecture, one has in hand the tower construction of Jones that allows analysis of the index of subfactors in Lecture 4 and the representations of the braid group that lead to the construction of the Jones polynomial in Lecture 7.

Lecture 3 gives a proof of Jones's theorem on the spectrum of values for the index of inclusions of type II_1 factors with the same identity. The theorem states that, given $N \subset M$, both II_1 factors with the same identity, then $[M : N] < 4$ implies the existence of $n \geq 3$ such that $[M : N] = 4\cos 2(\pi/n)$. For R the hyperfinite type II_1 factor and for each integer $n \geq 3$ there is a subfactor R_0 of R with $[R : R_0] = 4\cos 2(\pi/n)$, and for any real $r \geq 4$ there is a subfactor R_0 of R with $[R : R_0] = r$.

Lecture 3 proves Jones's theorem and compares its proof with the theorem of Friedan, Qui, and Shenker about the unitary representations of the Virasoro algebra. These two theorems have a remarkable similarity that suggests (along with mathematical evidence from statistical mechanics) an underlying relationship between the Temperley-Lieb algebra and the Virasoro algebra.

Lecture 4 introduces Bratelli diagrams, specific examples, and problems about indices of factors (the relative communtant problem). Lecture 5 introduces the Artin braid group and its representations (Burau, Gassner, Temperley-Lieb, Yang-Baxter equation, hints of quantum field theory). Lecture 6 concerns knots and links. It introduces the fundamental group of a link complement, the Alexander module, Seifert surfaces, Seifert matrices and S-equivalence, Alexander's Theorem (every knot or link can be represented by a closed braid), and the Markov Theorem that shows how to get (in principle) invariants of knots and links via a generalized trace on a representation of the Artin braid group.

Lecture 7 constructs the original Jones polynomial $V_L(t)$ via a Markov trace on a representation of the braid group to the Temperley-Lieb algebra. The lecture then goes on to discuss skein relations and the Kauffman bracket state model for $V_L(t)$. This involves a diagrammatic interpretation of the Temperley-Lieb algebra. The lecture includes brief discussions of the other skein polynomials (Homfly(pt), Lickorish-Millett-Ho, Kauffman) and the work of Kauffman, Murasugi, and Thistlethwaite on the Tait conjectures for alternating links.

Lecture 8 discusses knots and statistical mechanics with invariants arising from partition functions defined on link diagrams (of which the bracket model for $V_L(t)$ is the first and simplest example). Lecture 9 takes up the algebraic approach to constructing the Markov trace on a Hecke algebra (due to Adrian Ocneanu), the work of Morton and Short, the work of Wenzl and the Birman-Murakami-Wenzl algebra, and Wenzl's result of Brauer's centralizer algebra. It ends with a discussion of quantum invariant theory.

This book faithfully documents the state of the art in this field circa 1987—just before the breakthroughs of Witten and Atiyah [27, 2] (quantum field theory, topological quantum field theory) and Reshetikhin and Turaev [25] (application of quantum groups to construct link invariants and three-manifold invariants) and many more things that have come and are yet to come.

References

1. J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **20** (1923), 275–306.
2. M. F. Atiyah, *The geometry and physics of knots*, Cambridge Univ. Press, London, 1990.
3. R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York, 1982.
4. J. S. Birman, *Braids, links and mapping class groups*, Ann. of Math. Stud., vol. 92, Princeton Univ. Press, Princeton, NJ, 1974.
5. J. H. Conway, *An enumeration of knots and links and some of their algebraic properties*, Computational Problems in Abstract Algebra, Pergammon Press, New York, 1970, pp. 329–358.
6. R. H. Fox, *A quick trip through knot theory*, Topology of Manifolds (M. K. Fort, ed.), Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 120–167.
7. R. H. Fox and J. W. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math. **3** (1966), 257–267.
8. P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, C. K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 239–246.
9. V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.
10. ———, *A polynomial invariant for links via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **129** (1987), 103–112.
11. ———, *A new knot polynomial and von Neumann algebras*, Notices of Amer. Math. Soc. **33** (1986), 219–225.
12. ———, *On knots invariants related to some statistical mechanics models*, Pacific J. Math. **137** (1989), 311–334.
13. L. H. Kauffman, *The Conway polynomial*, Topology **20** (1980), 101–108.
14. ———, *Formal knot theory*, Math. Notes, vol 30, Princeton Univ. Press, Princeton, NJ, 1983.
15. ———, *On knots*, Ann. of Math. Stud., vol. 115, Princeton Univ. Press, Princeton, NJ, 1987.
16. ———, *Knots and physics*, World Scientific Press, Singapore, 1991, 1993.
17. ———, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.
18. ———, *Statistical mechanics and the Jones polynomial*, Proceedings of the 1986 Santa Cruz Conference of the Artin Braid Group, Contemp. Math., vol. 78, Amer. Math. Soc., Providence, RI, pp. 263–297.
19. ———, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.
20. L. H. Kauffman and S. Lins, *Temperley Lieb recoupling theory and invariants of 3-manifolds*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1994 (to appear).
21. J. W. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1968.
22. ———, *Infinite cyclic coverings*, Topology of Manifolds (Michigan State University 1967), Prindle, Weber, and Schmidt, Boston, 1968.

23. Przytycki and Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1987), 115–139.

24. K. Reidemeister, *Knotentheorie*, Chelsea Publ. Co., New York, 1984; Julius Springer, Berlin, 1932.

25. N. Y. Reshetikhin and V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

26. D. Rolfson, *Knots and links*, Publish or Perish Press, Cambridge, MA, 1976.

27. E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

Louis H. Kauffman
University of Illinois at Chicago