AN ISOPERIMETRIC INEQUALITY RELATED TO THUE'S EQUATION

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ABSTRACT. This paper announces the discovery of an isoperimetric inequality for the area of plane regions defined by binary forms. This result has been applied subsequently in the enumeration of solutions to the Thue inequality and, given its fundamental nature, may find application in other areas as well.

1. INTRODUCTION

Let $F(X, Y) = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n$ be a binary form with rational integer coefficients, and let A_F denote the area of the region $|F(x, y)| \leq 1$ in the real affine plane. In a seminal paper of 1909, Thue [11] showed that if F is an irreducible form with degree at least three and if h is a non-zero integer, then the equation $F(x, y) = h$ has only a finite number of solutions in integers x and y. In 1934, Mahler [7] gave an estimate for the number, $N_F(h)$, of solutions of the Thue inequality $|F(x, y)| \leq h$ in terms of the area, $A_F h^{2/n}$, of this region; in particular, he showed, under the same conditions as Thue, that

$$|N_F(h) - A_F h^{2/n}| \leq c h^{1/(n-1)}$$

where c is a number which depends only on F. Mahler left c and A_F unspecified, but did show that A_F is finite (under the above conditions).

Surprisingly little attention has been given to the estimation of A_F, even though it is clear that sharp estimates for A_F would have significant implications for Thue inequalities. Mueller and Schmidt [8,9] showed, under certain technical conditions, that A_F is bounded when the number of coefficients of F is small, but their approach gave little insight into the nature of A_F in general. The purpose of this paper is to announce the discovery of an isoperimetric inequality for A_F. Detailed proofs will be forthcoming in [4].

Theorem 1. *Let F be a binary form with integer coefficients having non-zero discriminant and degree at least three. Then*

$$A_F \leq 3B \left(\frac{1}{3}, \frac{1}{3} \right)$$

where $B(\frac{1}{3}, \frac{1}{3})$ denotes the Beta function with arguments of $\frac{1}{3}$. This bound is attained for forms with integer coefficients which are equivalent under $\mathrm{GL}_2(\mathbb{Z})$ to $XY(X - Y)$.
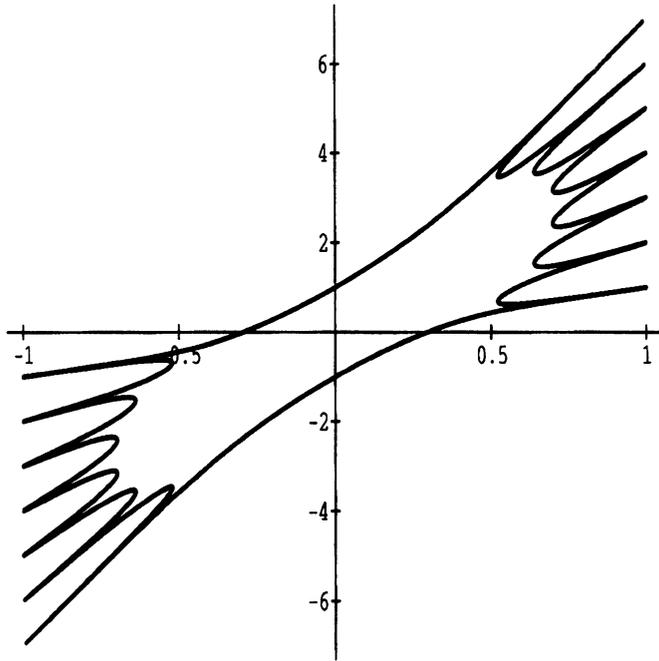
The approximate numerical value of $3B(\frac{1}{3}, \frac{1}{3})$ is 15.8997. A similar result is true for forms with complex coefficients provided that one includes a factor

FIGURE 1. $|P_7(x, y)| = 1$

involving the discriminant of F (see §2). However, the conditions on the discriminant and the degree cannot be relaxed in general since, for example, the forms X^n and $X^2 - Y^2$ give rise to infinite area. The condition that F have non-zero discriminant is equivalent to the condition that neither $F(x, 1)$ nor $F(1, y)$ have multiple roots and is satisfied, in particular, when F is an irreducible form. The notion of equivalence under $GL_2(\mathbb{Z})$ will be explained in §2.

One of the striking aspects of Theorem 1 is that the bound given for A_F is relatively small. In light of Mahler's result, one might expect $N_F(1)$ to be bounded and small as well. However, an examination of the forms

$$P_k(X, Y) = X^{2k} + (X - Y)^2(2X - Y)^2 \cdots (kX - Y)^2$$

reveals that this is not the case. Indeed, the region $|P_k(x, y)| \leq 1$ contains at least $2k$ integer lattice points, namely,

$$\pm(1, 1), \pm(1, 2), \dots, \pm(1, k),$$

but nevertheless is bounded (since P_k is positive definite) and has area less than 16. (See Figure 1 for the graph of P_7.)

These examples suggest that a finer analysis of the error term in Mahler's result could be quite delicate. However, they raise the interesting question of classifying forms according to the size of this error.

2. THE MAIN IDEAS

There are three main ideas involved in the proof of Theorem 1. The first and most important is to consider the estimation of A_F over the larger class of forms having complex coefficients.

For any form F, let D_F denote the discriminant of F. If F has a factorization $\prod_{i=1}^{n}(\alpha_i X - \beta_i Y)$ with α_i, $\beta_i \in \mathbb{C}$ (every binary form with complex coefficients has such a factorization), then $D_F = \prod_{i<j}(\alpha_i \beta_j - \alpha_j \beta_i)^2$. Let $\mathrm{GL}_2(\mathbb{R})$ denote the group of 2×2 real invertible matrices. For each $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$, let $F_T(X, Y) = F(aX + bY, cX + dY)$. Two forms are said to be equivalent under $\mathrm{GL}_2(\mathbb{R})$ if $G = F_T$ for some $T \in \mathrm{GL}_2(\mathbb{R})$. Equivalence under $\mathrm{GL}_2(\mathbb{Z})$ is defined similarly.

In this more general setting, we can prove the following inequality.

Theorem 2. *Let F be a binary form with complex coefficients having degree $n \geq 3$ and discriminant $D_F \neq 0$. Then*

$$|D_F|^{1/n(n-1)} A_F \leq 3B\left(\frac{1}{3}, \frac{1}{3}\right).$$

This bound is attained precisely when F is a cubic form which, up to multiplication by a complex number, is equivalent under $\mathrm{GL}_2(\mathbb{R})$ to the form $XY(X - Y)$.

Since the discriminant of a form with integer coefficients is an integer, Theorem 1 is an immediate consequence. The inequality in Theorem 2 is an *isoperimetric* inequality since it is sharp and connects the geometric quantities A_F and $|D_F|$ which are related to the shape and size of the region $|F(x, y)| \leq 1$. The quantity $|D_F|^{1/n(n-1)} A_F$ is natural to consider since it is invariant under $\mathrm{GL}_2(\mathbb{R})$ (i.e., $|D_{F_T}|^{1/n(n-1)} A_{F_T} = |D_F|^{1/n(n-1)} A_F$ for all $T \in \mathrm{GL}_2(\mathbb{R})$) in the same way that A_F is invariant under $\mathrm{GL}_2(\mathbb{Z})$.

The real advantage of considering $|D_F|^{1/n(n-1)} A_F$ instead of A_F is that one can apply the inequality between arithmetic and geometric means in a systematic fashion to reduce the estimation of $|D_F|^{1/n(n-1)} A_F$ to cubic forms. This is the second main idea. To be specific, let

$$M_n = \max |D_F|^{1/n(n-1)} A_F$$

where the maximum is taken over all forms of degree n with $D_F \neq 0$. Then one can show that $\{M_n\}$ is a decreasing sequence.

The third main idea is to consider the quantity $|D_F|^{1/n(n-1)} A_F$ as a function of n complex variables (the roots of $F(x, 1)$). By appealing to an appropriate maximum principle, one can then deduce that $|D_F|^{1/n(n-1)} A_F$ is maximized by a form F for which $F(x, 1)$ has n distinct real roots. Heuristically, one expects $|D_F|^{1/n(n-1)} A_F$ to be maximized by a form whose corresponding graph $|F(x, y)| = 1$ has the maximal number of asymptotes in the real affine plane (i.e. by a form for which $F(x, 1)$ has n distinct real roots). In the case of cubics, every such form is equivalent under $\mathrm{GL}_2(\mathbb{R})$ to $XY(X - Y)$.

3. Related questions

The outline given above raises some interesting questions regarding the sequence $\{M_n\}$. In particular, one might like to know all the values of this sequence and its limiting behavior. These and other questions will be addressed in [3]. However, based on that work, we believe that the following is true.

Conjecture. The maximal value M_n of the quantity $|D_F|^{1/n(n-1)} A_F$ over the class of forms of degree n with complex coefficients and non-zero discriminant

is attained precisely when F is a form which, up to multiplication by a complex number, is equivalent under $\mathrm{GL}_2(\mathbb{R})$ to the form

$$F_n^*(X, Y) = \prod_{k=1}^{n} \left(X \sin\left(\frac{k\pi}{n}\right) - Y \cos\left(\frac{k\pi}{n}\right) \right).$$

Moreover, the sequence of maximal values $\{M_n\}$ decreases monotonically to the value 2π.

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