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Classes of linear operators, Volume 2, by I. Gohberg, S. Goldberg, and M. A. Kaashoek, Operator Theory: Advances and Applications, vol. 63, Birkhäuser, Basel, 1993, x + 646 pp., $155.00. ISBN 3-7643-2944-0

This sixty-third volume of the Birkhäuser series in Operator Theory: Advances and Applications returns to the fundamentals of the first volume by H. Bart, I. Gohberg, and M. A. Kaashoek on *Minimal factorization of matrix and operator functions.* Operator theory is the field of mathematics that has the strongest interaction with the scientific and technological developments which are characteristic of the twentieth century. The present volume is an essential definition of the field for any center of learning that would like to offer viable programs in contemporary mathematics and engineering.

Operator theory is at the same time an abstract discipline which has the invariant subspace concept as a central theme. Although a continuous linear transformation of a nontrivial Banach space into itself need not have a nontrivial invariant subspace, it is expected that a nontrivial invariant subspace always does exist for the adjoint transformation, which acts on the dual Banach space. Should this expectation not be satisfied, there still remains a plentiful supply of operators that do have invariant subspaces. The field of operator theory can safely concentrate its efforts on the applications of invariant subspaces when they exist. The results obtained, however, acquire new significance as more general existence theorems for invariant subspaces are found.

The present volume begins with a discussion of the integral representation of an operator that can sometimes be given when a maximal chain of invariant subspaces is known. Projections onto the invariant subspaces are assumed given, and a uniformity hypothesis is made. These conditions are satisfied, for example, by a compact operator acting on a Hilbert space. The operator is represented as a Stieltjes integral using the projections belonging to the chain.

Such a chain of projections can also be applied to operators which do not have the ranges of the projections as invariant subspaces. The objective is then to factor the operator into a product of an operator which does have the ranges of the projections as invariant subspaces and an operator which has the ranges of complementary projections as invariant subspaces. In finite-dimensional spaces this corresponds to a factorization of a matrix as the product of a lower triangular matrix and an upper triangular matrix. There is an unavoidable ambiguity in such factorizations because of the existence of diagonal matrices. The treatment of the diagonal becomes a fundamental issue in the integral representations which generalize the lower-upper triangular decomposition to infinite-dimensional spaces.

The transition that is made from matrices to linear transformations in spaces of infinite dimensions is a particularly fine aspect of the present work. Rarely do mathematicians employ such constructive methods that are rich in applications and are suggestive of theoretical generalization. The effect of these results is to reduce the theory of integral equations, and therefore also the spectral theory of differential equations, to invariant subspace theory.

Some comments need to be made on the technical aspects of the work since the present formulation does not contain the spectral theory of the Löwner differential equation as it appears in the proof of the Bieberbach conjecture. For this application the concept of an invariant subspace needs to be generalized. Spaces which are contained continuously and contractively but not necessarily isometrically in the given space need to be considered. The appropriate generalization of a projection is a selfadjoint transformation P which satisfies the inequality $P^2 \leq P$ rather than the corresponding equality. Krein spaces seem to be the natural context for the formulation of integral representations because they permit a natural generalization of the concept of orthogonal complement which underlies the concept of projection.

A Krein space is a vector space with scalar product which is considered in the weak topology induced by duality with itself. The space also has a Mackey topology which is assumed computable by a Hilbert space metric. Continuity of a transformation is in principle taken with respect to the weak topology, but an equivalent concept of continuity for linear transformations is obtained with respect to the metric topology. A linear transformation T of a Krein space $\mathscr{P}$ into a Krein space $\mathscr{Q}$ is said to be contractive if the inequality

$$\langle Tc, Tc\rangle_{\mathscr{Q}} \leq \langle c, c\rangle_{\mathscr{P}}$$

holds for all elements c of $\mathscr{P}$.

If a Krein space $\mathscr{P}$ is contained continuously and contractively in a Krein space $\mathscr{H}$, then a unique Krein space $\mathscr{Q}$ exists which is contained continuously and contractively in $\mathscr{H}$ and has these properties: The inequality

$$\langle c, c\rangle_{\mathscr{H}} \leq \langle a, a\rangle_{\mathscr{P}} + \langle b, b\rangle_{\mathscr{Q}}$$

holds whenever $c = a + b$ with a in $\mathscr{P}$ and b in $\mathscr{Q}$. Every element c of $\mathscr{H}$ admits some such decomposition for which equality holds.

The space $\mathscr{Q}$ so obtained is uniquely determined by the space $\mathscr{P}$ and is called the complementary space to $\mathscr{P}$ in $\mathscr{H}$. Minimal decomposition is unique. It is obtained with $a = Pc$ and $b = (1 - P)c$ where P is a selfadjoint transformation of $\mathscr{H}$ into itself whose graph coincides with the adjoint of the inclusion of $\mathscr{P}$ in $\mathscr{H}$. The transformation P so obtained satisfies the inequality $P^2 \leq P$. If P is a given selfadjoint transformation of a Krein space $\mathscr{H}$ into itself which satisfies the inequality $P^2 \leq P$, then unique complementary Krein spaces $\mathscr{P}$ and $\mathscr{Q}$ exist such that the above computation of minimal decompositions applies.

The concept of complementation differs from the usual concept of orthogonal complement only in that nonzero elements may exist in the intersection of $\mathscr{P}$ and $\mathscr{Q}$. These elements however are well behaved in that they form a Hilbert space $\mathscr{L}$ with scalar product defined by

$$\langle c, c\rangle_{\mathscr{L}} = \langle c, c\rangle_{\mathscr{P}} + \langle c, c\rangle_{\mathscr{Q}}.$$

The existence of the Hilbert space effectively dilates complementation theory to the usual concept of orthogonal complement. Theorems about projections, and therefore the integral representations of the present volume, have an immediate generalization with selfadjoint transformations P which satisfy the inequality $P^2 \leq P$.

The generalization is significant because the notion of subspace carries with it a notion of size rather than just a notion of space occupied. The notion of size is flexible in that negative quantities are permitted, as in borrowing money. The resulting estimation theory is essential in interpolation problems related to the Bieberbach conjecture.

The theory of integral representations is the fifth part of the present two-volume series. The sixth part is concerned with Toeplitz operators.

One of the characteristic features of twentieth century mathematics is the remarkable interplay between invariant subspace theory and complex analysis. The concept of a Toeplitz operator allows the most immediate expression of that important relationship.

If m is a given positive integer, consider the m-dimensional Euclidean space realized as column vectors with complex entries. The conjugate transpose of such a vector c is a row vector c^-. The matrix product c^-c is a number which is equal to the square of the Euclidean norm of c. The vector Hardy space is defined as the Hilbert space of square summable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with vector coefficients

$$\|f\|^2 = \Sigma\, a_n^- a_n.$$

There is a related Hilbert space of vector Laurent series.

An $m \times m$ matrix with complex entries is regarded as an operator on vectors using the usual matrix multiplication. A more complicated operator is associated with a Laurent series

$$\Phi(z) = \sum_{-\infty}^{+\infty} A_n z^n$$

whose coefficients are such matrices. If

$$f(z) = \sum_{-\infty}^{+\infty} a_n z^n$$

is a square summable vector Laurent series, then a square summable vector Laurent series

$$g(z) = \sum_{-\infty}^{+\infty} b_n z^n$$

may be definable as a formal product

$$g(z) = \Phi(z) f(z).$$

This is the case when $\Phi(z)$ is the Fourier series expansion of a bounded measurable matrix function on the unit circle. Important special cases occur when the function represented on the unit circle is continuous or is in the Wiener algebra.

An algebra of operators on the space of square summable vector Laurent series is obtained. The adjoint of an operator in the algebra is again in the algebra. In the case that $\Phi(z)$ is a power series, an invariant subspace is given for every integer r as the set of those $f(z)$ such that $z^{-r} f(z)$ belongs to the vector Hardy space.

These examples of operators with invariant subspaces provide an opportunity to apply the integral representations of the previous chapters. A Wiener-Hopf factorization of the symbol $\Phi(z)$ is obtained when $\Phi(z)$ is not a power series. Of particular interest is the case in which $\Phi(z)$ is a rational matrix function, in which case the power series determined by the factorization are matrix polynomials.

The seventh part of this two-volume series is concerned with unitary linear systems and their transfer functions. The concept of a linear system has already been used in the treatment of Toeplitz operators.

For the purposes of the review a linear system is a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ whose entries are linear transformations acting on the Cartesian product of a vector space called the state space and a vector space called the external space. This is a simplification of the work reviewed where more general input–output systems with two external spaces are considered. Essential features of the theory are already found in the special case in which the two external spaces are equal.

The matrix acts on a space of column vectors whose upper entry is in the state space and whose lower entry is in the external space. The main transformation A maps the state space into itself. The input transformation B maps the external space into the state space. The output transformation C maps the state space into the external space. The external operator D maps the external space into itself. The transfer function of the linear system is the power series

$$W(z) = \sum_{n=0}^{\infty} W_n z^n ,$$

whose coefficients are the operators on the external space defined by $W_0 = D$ and

$$W_{n+1} = CA^n B$$

for every nonnegative integer n.

The present treatment of linear systems emphasizes the case in which the external space is the m-dimensional Euclidean space of column vectors, in which case the transfer function is a power series with $m \times m$ matrices as coefficients. Linear systems which have finite-dimensional state space are considered, in which case the transfer function is a rational matrix function. And linear systems are also considered whose state space is a Hilbert space of arbitrary dimension, in which case the external space is also allowed to be a Hilbert space of arbitrary dimension. The entries of the matrix of the linear system are required to be continuous transformations when a Hilbert space structure is present to supply topology. The matrix is also required to be unitary in that case.

The theory of linear systems, which is so presented, is all the more interesting for being incomplete. One theory applies when the state space has finite dimension, and the other theory applies when the state space is a Hilbert space. The overlap between these two theories is small, because unitary structure is a restrictive condition on the the transfer function when the state space is a finite-dimensional Hilbert space. An interesting direction for further research is indicated. Consider unitary linear systems whose state space is a Krein space instead of a Hilbert space. The problem is to determine whether such a theory can contain the theory of linear systems with finite-dimensional state spaces. It is too much to require that the unitary linear system has a finite-dimensional state space even in the case of indefinite scalar products. But it is reasonable to conjecture that a unitary linear system which is a dilation of the linear system with finite-dimensional state space exists.

A related problem is to construct a unitary linear system which has a given power series with $m \times m$ matrix coefficients as transfer function when the external space is an m-dimensional Euclidean space. Such a construction is now made when the power series defines a continuous and contractive multiplication in the Hardy space of square summable vector power series. But the construction fails to apply when the power series defines a continuous multiplication in the space which is not contractive. The problem of constructing unitary linear systems with given transfer functions therefore remains of interest for further work. Of particular interest is the construction of unitary linear systems whose transfer function is a given rational matrix function.

Linear systems are interesting because they can be multiplied when they have the same external space. The product of two linear systems is a linear system whose transfer function is the product of the transfer functions of the factors and which is a unitary linear system when the factors are unitary linear systems. The state space of the product linear system is now taken to be the orthogonal sum of the state spaces of the factors. The main transformation of the product linear system has the state space of the left factor as an invariant subspace and coincides with the main transformation of the left factor in the subspace. The resulting relationship between factorization and invariant subspaces is the reason for interest in linear systems.

A unitary linear system with main transformation A, input transformation B, and output transformation C is said to be pure when no nonzero element f of the state space exists such that $CA^n f$ and $B^* A^{*n} f$ are the zero element of the external space for every nonnegative integer n. Since unitary linear systems which are determined by their transfer functions are wanted, a desire for pure unitary linear systems results. If a given unitary linear system is not pure, there may exist a related pure unitary linear system which has the same transfer function. This is always the case when the set of such elements f forms a Hilbert space which is contained continuously and isometrically in the state space.

Complications in the relationship between factorization and invariant subspaces result because the product of two pure unitary linear systems need not be pure. However a related pure unitary linear system can often be constructed. Complementation theory supplies a context in which such a construction can be made. The decomposition of the state space of the product linear system into the state spaces of the factors is then made using a sum in the sense of complementation theory rather than an orthogonal sum. An alternative conception of the multiplication of unitary linear systems which is advantageous in applications results.

Applications of the theory of unitary linear systems appear in the interpolation theory of analytic functions. A problem which is due to Carathéodory and Schur is to characterize an initial segment of coefficients of a power series which represents a function which is bounded by one in the unit disk. A problem which is due to Bieberbach is to characterize an initial segment of coefficients of a power series which represents an injective function in the unit disk. A variant of the Bieberbach problem is to characterize an initial segment of coefficients of a power series which represents a function which is bounded by one and injective in the unit disk.

A solution of the Carathéodory-Schur problem is given using the theory of unitary linear systems whose state space is a Hilbert space and whose external space is the one-dimensional Hilbert space of complex numbers considered with absolute value as norm. A related construction which applies to the Bieberbach problem is the Grunsky transformation. A solution of the Bieberbach problem is not known because the extension problem which appears in that case is more difficult. For that purpose the treatment of extension problems, such as occur in commutant lifting, needs to be simplified. An extension problem is solved by constructing a norm which correctly describes the constraints on the extension to be made. The reviewer proposes complementation theory as a natural method of constructing such norms.

A diagram from complementation theory is typical of such constructions. Assume that $\mathscr{P}$ and $\mathscr{Q}$ are complementary subspaces of a Krein space $\mathscr{H}$ and that π is a contractive partial isometry of $\mathscr{H}$ onto a Krein space $\mathscr{H}'$. Then unique complementary subspaces $\mathscr{P}'$ and $\mathscr{Q}'$ of $\mathscr{H}'$ exist such that π acts as a contractive partial isometry of $\mathscr{P}$ onto $\mathscr{P}'$ and of $\mathscr{Q}$ onto $\mathscr{Q}'$. The diagram expresses a stability property of complementation under certain kinds of transformations. A Krein space which is decomposed, perhaps orthogonally, into complementary Krein spaces is given. Now hit the full space with a contractive partial isometry. Despite the severity of this distortion, the initial decomposition survives in a modified form which is expressed by the subtleties of complementation theory. This feature of complementation theory is what makes it useful in applications. Note that the underlying scalar products can be indefinite. Extension theorems which are formulated in terms of complementation theory therefore have an immediate generalization to Krein spaces.

An early application of complementation theory appears in the Douglas factorization lemma. Most discussions of the factorization, including the present one, have deleted the original argument related to complementation theory. An instructive exercise for the reader is to reconstruct the missing spaces, which are contained continuously but not isometrically in the full space. The factorization results from the contractive inclusion of one space in the other.

The eighth part of the present volume is a welcome treatment of the theory of Banach algebras. This subject is appropriate because some aspects of invariant subspace theory are best formulated for operator algebras rather than for isolated operators. The Gelfand theory of commutative Banach algebras is, for example, the main motivation for the existence of invariant subspaces. The Stone-Weierstrass theorem is a fundamental construction of invariant subspaces in that context.

The reviewer however was disappointed that the original proof of Marshall Stone was presented rather than the more direct argument which results when the Krein-Milman theorem is applied. There is a scarcity of applications of the Krein-Milman theorem, because few compact convex sets are known whose extreme points have interesting properties. The Stone-Weierstrass theorem supplies a context in which extreme points are characterized in a way which is useful for solving an approximation problem. Since the problem is also an invariant subspace problem, the Krein-Milman theorem becomes a principal motivation for the existence of invariant subspaces. Of particular interest is the relationship between invariant subspaces and approximation which is established in the Stone-Weierstrass context.

A successful invariant subspace theory results for a normal operator on a Hilbert space. A von Neumann algebra of operators on a Hilbert space is determined by a knowledge of the common invariant subspaces of the elements of the algebra, but a corresponding structure theory is not known for a more general algebra of operators on a Banach space. The adjoints of the elements of the algebra are operators on the dual Banach space. Metrically closed subspaces of the dual space which are common invariant subspaces for the adjoints of the elements of the algebra are expected. The algebra is expected to be essentially determined by a knowledge of such subspaces. Vector generalizations of the Stone-Weierstrass theorem are the sources of these expectations, which are satisfied in the case of a von Neumann algebra of operators acting on a Hilbert space.

Operator algebras are also considered for an axiomatization of Wiener-Hopf factorization. A Banach algebra with unit is said to be a decomposing algebra if it is the sum of two nontrivial closed subalgebras. Every element of the full algebra which is sufficiently close to the identity is uniquely the product of two elements, each associated with a given subalgebra. Applications of the factorization to the matrix Wiener algebra are given. Another insight into the factorization theory of rational matrix functions results.

The factorization theory so obtained has an undetermined relationship to the factorization theory for unitary linear systems. A factorization problem can be considered for unitary linear systems whose state space is a Krein space and whose external space is a finite-dimensional Hilbert space. The problem is to factor a given unitary linear system as a product of a unitary linear system whose state space is a Hilbert space and a unitary linear system whose state space is the antispace of a Hilbert space. Such a factorization is supplied by the Nevanlinna factorization theory for scalar functions which are analytic and of bounded type in the unit disk. A solution of the problem is expected when the transfer function defines a continuous transformation in the space of square summable vector Laurent series. Good unitary linear systems admit such factorizations. A construction of unitary linear systems which has these good features of the scalar theory is wanted. The reviewer would like to know whether the present Wiener-Hopf factorization can be applied to this fundamental factorization problem for unitary linear systems.

The final, ninth part of the volume is essentially an application of the previous methods to extension and completion problems. A technique called the band method is introduced to keep track of the consequences of the invariant subspace concept in extensions. There are two essentially equivalent ways of formulating the basic extension problem. In one formulation a contractive 2×2 matrix is to be constructed by suitable choice of the lower-diagonal entry when the other three entries are given. In the other formulation a positive 3×3 matrix is to be constructed by suitable choice of the off-diagonal corner entries when the other seven entries are given. In either formulation of the problem a parameterization of solutions is wanted.

The band method applies to the extension problem for selfadjoint matrices and generalizes it to an abstract context. A particular extension, called the central or band extension, is found, which is characterized by a maximum entropy property. All other extensions are then derived from the band extension using a Wiener-Hopf factorization formulated in a conjugated generalization of the theory of decomposing algebras. A linear fractional parameterization of extensions results.

Some questions are left unanswered by this elegant construction. If a linear fractional transformation is present, then its defining matrix is related to a linear system. If the theory of unitary linear systems is to be taken seriously, then the construction needs to be supplied with scalar products which make the linear system unitary. When a unitary linear system has been found, the linear fractional transformation needs to be seen as a factorization in the theory of unitary linear systems.

Instructive applications of the present factorization theory are given to the Carathéodory-Toeplitz extension problem, the Nehari extension problem, and the Nevanlinna-Pick interpolation problem. The additional structure of a unitary linear system is present in all these examples. They suggest that the present band method can be restructured as a construction of unitary linear systems.

The authors are to be congratulated for an instructive formulation of the current status of a field which has a major impact on contemporary science and technology. Although the theories are not in final form, the methods applied are permanent because they are algorithms of computation. Further research can only deepen the understanding of why these methods are successful and widen the scope of their applications.

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Nilpotence and periodicity in stable homotopy theory, by Douglas C. Ravenel. Princeton University Press, Princeton, NJ, 1992, xiv + 209 pp., $24.95. ISBN 0-691-02572-X

The subject of homotopy theory, especially stable homotopy theory, was for many years guided by J. Frank Adams. In the final article in his selected works [1] he wrote: "At one time it seemed as if homotopy theory was utterly without system; now it is almost proved that systematic effects predominate." Adams was commenting on the influence of the results discussed in Ravenel's book, which are the subject of this review. The most striking of these results are due to Ethan Devinatz, Mike Hopkins, and Jeff Smith [2, 5] and were conjectured by Doug Ravenel [7] in the late seventies and early eighties.

To set the stage, recall that two continuous maps f and g from a space X to a space Y are *homotopic* if there is a continuous map $H : X \times [0, 1] \to Y$ agreeing with f on $X \times \{0\}$ and with g on $X \times \{1\}$. One often restricts attention to CW-complexes, i.e. spaces built in a systematic way by attaching cells. In stable homotopy theory, one is permitted to suspend a map $f : X \to Y$ as often as desired; its suspension $\Sigma f : \Sigma X \to \Sigma Y$ is defined in a natural way on the suspension of X, the "double cone" obtained from $X \times [0, 1]$ by collapsing