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BOOK REVIEW

Evolutionary integral equations and applications, by Jan Prüss, Monographs in Mathematics, vol. 87, Birkhäuser Verlag, Boston, MA, 1993, xxvi+366 pp., \$139.00. ISBN 3-7643-2876-2

This book deals with linear integral equations of the following type:

$$u(t) = \int_0^t A(t-s)u(s)\,ds + f(t), \qquad t \geq 0 \tag{1}$$

(and also with the corresponding equation on the line, see equation (3) below) where $u\colon [0,+\infty[\to X$ is a function with values in a Banach space X; $A(t)$, $t \geq 0$, are linear operators, generally unbounded; and $f\colon [0,+\infty[\to X$ is a given function.

There is an expansive literature regarding equation (1) when the space X is finite dimensional; see, for example, the monograph [3]. In infinite dimensions there are also several papers and, in addition, a monograph about nonlinear problems in viscoelasticity [6] but not, to our knowledge, a systematic treatise before the present book.

Equation (1) is studied by looking for a resolvent $S(t)$, $t \geq 0$, that is an operator-valued mapping such that (formally)

$$S(t) = \int_0^t A(t-s)S(s)\,ds + I, \qquad t \geq 0\,.$$

Once the resolvent is obtained, the solution to (1) is given by

$$u(t) = \frac{d}{dt}\int_0^t A(t-s)S(s)f(s)\,ds\,.$$

There are obviously several difficulties connected with this program, due to the lack of boundedness of operators $A(s)$. In the book, the case when $A(s) = K(s)A$, where $K(\cdot)$ is a scalar kernel, is first treated. Because it is obviously impossible to give a detailed description of all the results contained in the book, we quote, for instance (freely), the following generation result, which is a generalization of the classical Hille-Yosida Theorem for a strongly continuous semigroup of linear operators.

Theorem 1. *Let $A(t) = a(t)A$ with $A\colon D(A) \subset X \to X$ a closed, densely defined operator and a scalar Laplace transformable kernel, with $\int_0^{+\infty} e^{-\omega t}|a(t)|\,dt < +\infty$. Then there exists a resolvent $S(\cdot)$ for equation (1), if and only if, for $\operatorname{Re}\lambda > \omega$, the Laplace transform $\hat{a}(\lambda)$ is different from 0, $1/\hat{a}(\lambda)$ belongs to the resolvent set of A, and $H(\lambda) = 1/\lambda(I - \hat{a}(\lambda)A)$ fulfills the following estimates*:

$$\left\|\frac{d^n H}{d\lambda^n}\right\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}, \qquad n = 1, 2, \ldots; \ \operatorname{Re}\lambda > \omega\,. \tag{2}$$

Obviously it is not easy to check the infinitely many estimates of (2) in general. However, this can be done for instance if one of the following statements holds

(i) X is a Hilbert space, A is selfadjoint nonnegative, and a is a positive kernel, that is, $\operatorname{Re}\hat{a}(\lambda) \geq 0$ whenever $\operatorname{Re}\lambda \geq 0$.

(ii) A generates an analytic semigroup, and $\hat{a}(\lambda)$ is extendable in a sector $S_{\omega,\theta} = \{\lambda \in \mathbf{C} : \arg|\lambda - \omega| < \theta\}$, $\theta > \pi/2$, of the complex plane, and

$$\|H(\lambda)\| \leq \frac{C}{|\lambda - \theta|}, \qquad \lambda \in S,$$

for some constant $C > 0$. This is the so-called *parabolic case*.

(iii) A generates a contraction semigroup and a is a completely positive kernel; see [1].

Moreover, an important tool to prove existence of resolvents for other problems (both parabolic and hyperbolic) is the so-called *subordination principle*; see [5].

Among the other arguments treated in the book, let us recall

(i) generalisation of generation theorems to nonscalar kernels,
(ii) a new variational approach to integrodifferential equations, and
(iii) maximal regularity results in the parabolic case.

The last part of the book is devoted to integrodifferential equations on the line

$$u(t) = \int_{-\infty}^{t} A(t-s)u(s)\,ds + f(t), \qquad t \geq 0\,. \tag{3}$$

Equation (3) is studied in homogeneous spaces, that is, spaces of functions on the real line where translations are bounded operators; this is a natural requirement due to the translation invariance of equation (3). Homogeneous spaces include: spaces of uniformly continuous and bounded functions, L^p functions, and periodic and almost periodic functions. The main tool used here is an integrability property of the resolvent $S(t)$ that is studied in different situations.

The prerequisites to read the book are essentially: basic functional analysis and semigroup theory. The method used can be seen as a generalisation to that theory. For instance, if $a(t) = 1$ and $f(t) = x$, equation (1) reduces to

$$u'(t) = Au(t), \qquad u(0) = x; \tag{4}$$

and if $a(t) = t$ and $f(t) = x + ty$, equation (1) reduces to

$$u''(t) = Au(t), \qquad u(0) = x, \qquad u'(0) = y\,.$$

It is possible, under suitable conditions and by an appropriate choice of the state spaces, to reduce a general equation (1) to an evolution equation of the form (4); see [4]. As pointed out by the author, this approach presents several disadvantages due to the arbitrariness of the choice of the state space, which looks somewhat artificial, and the lack of regularity properties of the previous system. For instance, starting from parabolic integrodifferential equations, one can find hyperbolic evolution equations. However, this second approach is very useful in the study of a linear quadratic control problem associated with problem (1) because it allows the use of dynamic programming; see [2]. The same happens if one is perturbing (1) by a white noise, since, in this form, the Markov property is lost.

In my opinion this book is written in a very clear and effective way. Moreover, several applications of the theory are extensively presented, with a careful description. In particular, we mention applications to: viscoelasticity, heat conduction in material with memory, thermo-viscoelasticity, and electrodynamics with memory.

In conclusion, the book is welcome, thermo-viscoelasticity and I am sure that it will be very useful for all scientists working in the field of linear integrodifferential equations.

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