APPEARED IN BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 2, October 1994, Pages 308-311

BOOK REVIEW

Contact geometry and linear differential equations, by V. Nazaikinskii, V. Shatalov, and B. Y. Sternin. Walter de Gruyter, Berlin, 1992, vii+216 pp., $69.00. ISBN 3-11-013381-4

During the first decades of this century, the investigations of asymptotic oscillatory solutions for the Schrödinger equation in quantum mechanics and for the wave equation in geometrical optics were mainly performed by using the W-K-B method and the method of characteristics. More precisely, the expected asymptotic solutions were of the form $e^{i\tau\varphi(x)}a(x)$ where φ is classically known as the phase function and a the amplitude function.

Since then, these methods have spread to the general theory of differential equations thanks to the research of many mathematicians mainly motivated by theoretical physics (see, for example, the works of Voros [8] and Sato, Kawai, and Kashiwara [7] and the references in the book under review).

The basic idea is to perform a "transformation" of the equations, which is also the idea of the classical Fourier transform (denoted by $\hat{}$): By defining $\hat{u}$ for u belonging to the space S of rapidly decreasing functions in $\mathbb{R}^n$ (the Schwartz space), one has

$$\hat{u}(\xi) = \int e^{-i\langle x,\xi\rangle} u(x)\, dx\,.$$

This formula defines an isomorphism from S onto S. Using the inverse of $\hat{}$, u can be recovered by the formula:

$$(*) \qquad u(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} u(y)\, dy\, d\xi\,.$$

A fundamental result in Fourier analysis is that $\hat{}$ extends to an isomorphism of the dual of S, denoted as S' (that is, the space of temperate distributions on $\mathbb{R}^n$), which turns out to be the smallest subspace of distributions containing L^1 and is also invariant by the action of linear partial differential operators with polynomial coefficients.

When P is such an operator, that is,

$$P(x, D_x) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha, \qquad \alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n,$$

$D_x^\alpha = D_{x_n}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ and $a_\alpha(x) \in \mathbb{C}[x_1, \dots, x_n]$, one may define $\widehat{P}(\xi, D_\xi)$, the Fourier transform of P, as the operator obtained by the transformation

$$\xi_i = i\frac{\partial}{\partial x_i}, \qquad \frac{\partial}{\partial \xi_i} = -x_i;$$

and one gets the relation

$$\widehat{Pu}(\zeta) = \widehat{P}\hat{u}(\zeta)\,.$$

Now, let $P(x, D_x)$ be a partial differential operator with C^∞-coefficients of the form

$$P(x, D_x) = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha\,.$$

Using $(*)$, one obtains

$$(**) \qquad Pu(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} \sigma(P)(y,\xi) u(y)\, dy\, d\xi$$

where $\sigma(P)$ denotes the total symbol of P, that is, the C^∞-function in (x, ξ)

$$\sigma(P)(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)(i\xi)^\alpha\,.$$

We call the principal symbol of P the nonvanishing homogeneous term of $\sigma(P)$ of highest degree. Its degree is also the order of P.

This description of the action of P by $(**)$ shows that any linear partial differential operator is actually a particular case of a pseudodifferential operator which theory was largely developed in the sixties and the seventies (see the works of Zygmund, Calderón, Kohn, Nirenberg, Lax, Hörmander—the references of the book under review). One of the main features of this theory is that elliptic differential operators (like the Laplacian or the Cauchy-Riemann operator) are invertible in the class of pseudodifferential operators.

More generally, one defines Fourier integral operators, that is, operators whose action may be described by integrals of the form

$$Pu(x) = \iint e^{i\varphi(x,y,\xi)} a(x,y,\xi) u(y)\, dy\, d\xi$$

where φ is real homogeneous of degree one with respect to the action of $\mathbb{R}_+$ and the amplitude a has an asymptotic expansion of the form

$$a(x,y,\xi) \sim \sum_{j=0}^{\infty} a_j(x,y,\xi),$$

a_j being C^∞-functions on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n - \{0\}$ homogeneous of degree $(\lambda - j)$ in ξ for a certain $\lambda \in \mathbb{R}$.

A theory very similar to this was developed by Maslov [9] in the sixties with the introduction of the "canonical operator", which is one of the subjects of this book.

If one wants to state an invariant theory, the natural framework is the cotangent vector bundle T^*X (the phase space) to the manifold X. The cotangent vector bundle has a particular geometric structure, that is, a homogeneous symplectic structure: if $x = (x_1, \ldots, x_n)$ is a system of local coordinates in X, (x, ξ) the associated coordinates in T^*X, the image of the canonical 1-form $\sum_{i=1}^n \xi_i \, dx_i$ by the Hamiltonian isomorphism associated to the (nondegenerate closed) 2-form $\sigma = d\omega$ is precisely $-\theta$, the radial vector field of T^*X.

In this situation the principal symbol of a linear partial differential operator turns out to be a well-defined homogeneous function on T^*X (with respect to the $\mathbb{R}_+$-action). These kinds of ideas still hold for Fourier integral operators and are the starting point for microlocal analysis.

Let us now point out that differential symplectic geometry is also the natural framework for classical mechanics, since one of its fundamental concepts is that of the Hamilton system, that is, first-order linear systems of the type:

$$\frac{\partial x}{\partial t} = \frac{\partial f}{\partial \zeta}$$

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial f}{\partial x}, \qquad f \text{ being the Hamilton function}\,.$$

Such questions are studied in this book from another point of view—the authors consider homogeneous objects with respect to the action of $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ on T^*X, so it is natural to consider the contact structure of the projective cotangent bundle $\mathbb{P}^*X = \mathring{T}^*X/\mathbb{R}^*$ where $\mathring{T}^*X$ denotes the symplectic homogeneous manifold T^*X with the zero section deleted.

To end this review, let us say a few words about microlocal analysis from the analytic point of view, that is, Sato-Kashiwara-Kawai's theory of microdifferential operators, which has been developing since the end of the sixties [10]. In this case one considers the homogeneity with respect to the action of $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ on T^*X (which is a complex homogeneous symplectic manifold).

Systems of linear differential equations are viewed as coherent modules over the sheaf of linear differential holomorphic operators—the already classic theory of $\mathcal{D}$-modules.

The analogue of Fourier integral operator is the operation called "quantization of canonical transformation".

Sheaf theory and derived categories are systematically used, since they give the full understanding of obstruction to properties of the solutions.

The above relation concerning Fourier transform for functions and differential operators with polynomial coefficients was generalized in the following terms: first, one defines an operation for complexes of sheaves on a vector bundle (viewed as objects of a derived category) which have homogeneous cohomology with respect to the action of $\mathbb{R}_+$. Similarly one defines Fourier transform for algebraic $\mathcal{D}$-modules, that is, modules over the sheaf of differential operators with polynomial coefficients with respect to the variables of the fiber. If the $\mathcal{D}$-module satisfies a certain condition of homogeneity, the sheaf-theoretic Fourier transform (also called Fourier-Sato transform) exchanges the "solutions" of the $\mathcal{D}$-module and the "solutions" of its transform. Here, by "solutions" we mean once more not only the solutions in the usual sense but also in the sense of derived categories (for further details we refer to the works by Katz, Laumon, Brylinski, Verdier, Malgrange, Hotta, and Kashiwara in the eighties).

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