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Nonlinear potential theory of degenerate elliptic equations, by J. Heinonen, T. Kilpeläinen, and O. Martio. Oxford University Press, London, 1993, v+363 pp., $70.00. ISBN 0-19-853669-0

This is a remarkable book which covers a period of three decades starting with Serrin's 1964 fundamental paper [29] on the Harnack inequality and Hölder continuity of solutions to quasilinear equations with nonquadratic growth in the gradient. In that paper Serrin was able to extend the celebrated theorem of De Giorgi–Nash–Moser to a general class of nonlinear second-order equations. The prototype of such equations arises as the Euler equation of the energy functional

$$D_p(u;\Omega) = \int_\Omega |\mathscr{D}u|^p\,dx\,, \qquad 1 < p < \infty. \tag{1}$$

Critical points of this functional are (weak) solutions to the so-called p-Laplacian

$$\Delta_p u = \operatorname{div}(|\mathscr{D}u|^{p-2}\mathscr{D}u) = 0. \tag{2}$$

When $p = 2$, one recovers Laplace's equation $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0$. Solutions of $\Delta u \leq 0$ are called superharmonic functions. Classical potential theory finds its roots in the physical interpretation of superharmonic functions as potentials of compactly supported measures (mass or charge distributions). It has evolved to its present monumental stature out of the pioneering investigations of Laplace, Gauss, Poisson, Green, and Riemann. The development of modern potential theory constitutes a tour de force in which ideas from analysis, geometry, topology, complex function theory, and probability have been brought together. This effort is well witnessed in some of the classical sources on the subject; see [31], [3], [1], [15], [10], [7], [13], [20], [9].

Nonlinear potential theory requires an even harder effort, as new ideas need to be invented to supplement the lack of those available in the linear case. Over the last thirty years the study of nonlinear partial differential equations modelled on (2) has undergone a formidable development. Correspondingly, a nonlinear potential theory has been created, and these two branches of analysis have walked hand-in-hand and have offered inspiration to one another. The book under review is an excellent witness of this development. Before we turn to its discussion, however, we will attempt to give a simplified account of linear potential theory. We hope, in this way, that the unfamiliar reader will be better able to appreciate the subtleties involved in the nonlinear one.

Harmonic functions constitute one of the most beautiful chapters of classical analysis. Given an open set $\Omega \subset \mathbb{R}^n$, a function $u \in C^2(\Omega)$ is called harmonic if $\Delta u = 0$ in Ω. Gauss's theorem states that if u is harmonic in Ω, then

$$u(x) = \fint_{\partial B_r(x)} u \, d\sigma = \fint_{B_r(x)} u \, dy \tag{3}$$

for every $x \in \Omega$ and $r > 0$ such that the closure of the ball $B_r(x)$ is contained in Ω. Hereafter, the notation $\fint_E f \, d\mu$ stands for $\mu(E)^{-1} \int_E f \, d\mu$. Conversely, if $u \in C(\Omega)$ and (3) holds, then a classical theorem of Koebe states that $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω. The mean value formula (3) is the cornerstone of classical potential theory. Immediate consequences of it are the strong maximum principle for harmonic functions, Harnack's inequality and convergence theorem, and Liouville theorems, just to name a few. One can also use (3) to prove in an elementary fashion an important regularity result due to Caccioppoli [5], Cimmino [6], and Weyl [32], namely, that distributional solutions of $\Delta u = 0$ in Ω are in fact C^∞ and solve $\Delta u = 0$ in the classical sense.

Superharmonic functions are characterized by the property that

$$\fint_{\partial B_r(x)} u \, d\sigma \leq u(x) \tag{4}$$

for every $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$. Furthermore, any $u \in C(\Omega)$ which satisfies (4) verifies the strong minimum principle. One of the central results in potential theory is Perron's generalized solution to the Dirichlet problem [26]:

$$\Delta u = 0 \text{ in } \Omega, \qquad u\,|_{\partial\Omega} = f.$$

In Perron's method, one must allow for a more general notion of superharmonic function. The following definition was introduced by F. Riesz in his seminal paper [28]. Given an open set $\Omega \subset \mathbb{R}^n$, a function $u : \Omega \to (-\infty, \infty]$ is called superharmonic if: (i) u is lower semicontinuous (l.s.c.) in Ω; (ii) $u \not\equiv \infty$ in any open connected component of Ω; (iii) u satisfies (4). Subharmonic functions are defined as the negative of superharmonic ones. It is a classical result that a function u is subharmonic in Ω if and only if $u \in L^1_{\text{loc}}(\Omega)$ and $\Delta u \geq 0$ in the sense of distributions. Perron's method can be described as follows. Given a bounded open set $\Omega \subset \mathbb{R}^n$, let $f : \partial\Omega \to \mathbb{R}$ be bounded. Define the upper class $\mathscr{U}_f$ and the lower class $\mathscr{L}_f$ relative to f as $\mathscr{U}_f = \left\{ u \text{ superharmonic in } \Omega \,\middle|\, \underline{\lim}_{\partial\Omega} u \geq f \right\}$, $\mathscr{L}_f = \left\{ u \text{ subharmonic in } \Omega \,\middle|\, \overline{\lim}_{\partial\Omega} u \leq f \right\}$. Perron's upper and lower solutions are respectively given by

$$\overline{H}_f = \inf_{u \in \mathscr{U}_f} u, \qquad \underline{H}_f = \sup_{u \in \mathscr{L}_f} u.$$

It is an astonishing fact, and this is the central result in Perron's construction, that $\overline{H}_f$ and $\underline{H}_f$ are harmonic in Ω. A function f is called *resolutive* if $\overline{H}_f \equiv \underline{H}_f$. When this is the case, the harmonic function $H_f = \overline{H}_f = \underline{H}_f$ is the generalized Perron solution to the Dirichlet problem. Perron's ideas were elaborated by Wiener [33], [34], [35], and perfected by Brelot [2], [4]. For this reason, the function H_f is nowadays called the Perron–Wiener–Brelot (PWB) solution to the Dirichlet problem. Wiener proved that every $f \in C(\partial\Omega)$ is resolutive. This result and the maximum principle allows one to conclude that, for any fixed $x \in \Omega$, $f \mapsto H_f(x)$ defines a positive, continuous linear functional on $C(\partial\Omega)$. By the Riesz's representation theorem there exists a unique positive Borel measure $d\omega^x$ on $\partial\Omega$ such that

$$H_f(x) = \int_{\partial\Omega} f(Q)\, d\omega^x(Q).$$

The measure $d\omega^x$ is called the *harmonic measure* relative to Ω evaluated at x. Clearly, $d\omega^x(\partial\Omega) = 1$ for every $x \in \Omega$. Furthermore, Harnack's inequality implies that $d\omega^x$ and $d\omega^y$ are mutually absolutely continuous for every $x, y \in \Omega$. Later on, Brelot was able to show that f is resolutive if and only if $f \in L^1(\partial\Omega; d\omega^x)$ for any $x \in \Omega$, where $d\omega^x$ denotes the harmonic measure evaluated at x.

It had been known that the PWB solution to the Dirichlet problem for continuous data in general does not need to be continuous up to the boundary. A simple example was provided by Zaremba in 1911. Soon after, in 1912, Lebesgue constructed a more discouraging one; see [15]. However, Poincaré and Zaremba had proved that for a domain satisfying the outer cone condition, the Dirichlet problem is solvable in the classical sense. In his celebrated 1924 paper [34], Wiener succeeded in giving a geometric characterization of the so-called regular boundary points. A point $x_0 \in \partial\Omega$ is called regular for the Dirichlet problem if $\lim_{x \to x_0} H_f(x) = f(x_0)$ for every $f \in C(\partial\Omega)$. Wiener proved that $x_0 \in \partial\Omega$ is regular if and only if

$$\int_0^1 \frac{\operatorname{cap}((\mathbb{R}^n \setminus \Omega) \cap B_r(x_0)\,;\, B_{2r}(x_0))}{\operatorname{cap}(B_r(x_0)\,;\, B_{2r}(x_0))} \frac{dr}{r} = \infty. \tag{5}$$

For a given condenser $(E\,;\Omega)$, we have denoted by $\operatorname{cap}(E\,;\Omega)$ the Newtonian capacity of the condenser. Roughly speaking, Wiener's criterion (5) says that for a point $x_0 \in \partial\Omega$ to be regular, the complement of Ω must not be too thin at x_0.

In 1929 Kellogg published his treatise *Foundations of potential theory* [17]. This beautiful book, still much worth reading, marks the end of an epoch in potential theory. At the time in which Kellogg wrote his book, new powerful functional analytic ideas (whose development had been made possible by Lebesgue's integration theory) were entering the picture. In this connection, the closely related notions of harmonic measure and capacity play an important role. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let E be a compact subset of Ω. Consider the class $\mathscr{F}(E\,,\Omega) = \{u \in C_0^\infty(\Omega) \mid u \geq 1 \text{ on } E\}$. The variational capacity of the condenser $(E\,;\Omega)$ is given by

$$\operatorname{cap}(E\,;\Omega) = \inf_{u \in \mathscr{F}(E\,,\Omega)} \int_\Omega |\mathscr{D}u|^2\,dx. \tag{6}$$

This notion is obviously related to the variational interpretation of harmonic functions as free minimizers of the Dirichlet integral given by (1) when $p = 2$.

It can be recognized that the above definition is equivalent to the classical Newtonian capacity defined via the fundamental solution of the Laplacian. In standard modern language a harmonic function u in Ω is an element of the Sobolev space $H^{1,2}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \langle \mathscr{D}u\,,\mathscr{D}\varphi\rangle\,dx = 0 \tag{7}$$

for every compactly supported $\varphi \in \overset{\circ}{H}{}^{1,2}(\Omega)$. Suppose that $u \in H^{1,2}(\Omega)$ is harmonic and that $v \in H^{1,2}(\Omega)$ is such that $u - v \in \overset{\circ}{H}{}^{1,2}(\Omega)$ (so that in a suitable weak sense v takes the value u on $\partial\Omega$). Letting $\varphi = u - v$ in (7), one easily recognizes that

$$\int_\Omega |\mathscr{D}u|^2\,dx \leq \int_\Omega |\mathscr{D}v|^2\,dx.$$

We conclude that among all functions $u \in H^{1,2}(\Omega)$ having the same "boundary values" on $\partial\Omega$, harmonic functions are those which have the least energy $D(u\,;\Omega)$. The variational approach has linked potential theory to the study of second-order elliptic partial differential equations. One can, in fact, consider instead of (7) the more general formulation

$$\int_\Omega \langle A(x)\mathscr{D}u\,,\mathscr{D}\varphi\rangle = 0\,, \tag{8}$$

where $\mathscr{A} : \Omega \to M_{n\times n}(\mathbb{R})$ is a matrix-valued function with bounded measurable entries. $\mathscr{A}$ is said to be uniformly elliptic if there exists $\lambda > 0$ such that $\lambda|\xi|^2 \leq \langle \mathscr{A}(x)\xi\,,\xi\rangle$, for every $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$. A function $u \in H^{1,2}_{\mathrm{loc}}(\Omega)$ is called a weak solution to the equation $\mathscr{L}u = \operatorname{div}(\mathscr{A}(x)\mathscr{D}u) = 0$ in Ω if (8) holds for every $\varphi \in \overset{\circ}{H}{}^{1,2}(\Omega)$. One of the culminating results of this century's mathematics is the celebrated theorem of De Giorgi [8], Nash [25], Moser [24], stating that weak solutions of $\mathscr{L}u = 0$ are locally Hölder continuous (after modification on a set of measure zero). Moser, in fact, also proved a Harnack type inequality for nonnegative solutions to $\mathscr{L}u = 0$. This result opened the way to the development of potential theory for linear equations with rough coefficients. In this respect a fundamental theorem of Littman, Stampacchia, and Weinberger [22] shows that a boundary point is regular for the Dirichlet problem for the operator $\mathscr{L}$ if and only if it is regular for Δ.

We have thus finally come to the subject of the book under review. In 1938 Sobolev [30] published his famous embedding theorem: $H^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, where $1 < p < n$ and $q > p$ is given by the equation $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. The energy functional involved in Sobolev's theorem is that given by (1). As pointed out above, the case $p = 2$ is deeply connected with classical potential theory, and this has already been discussed.

Critical points of (1) are (weak) solutions to (2), thereby called p-harmonic functions. The case $p = n$ in (1), (2) has a special significance because of its connection with the theory of quasiregular mappings. This link will be further discussed below. From a physical standpoint equation (2), or rather its generalizations, arises naturally, e.g., in the steady rectilinear motion of incompressible non-Newtonian fluids or in phenomena of phase transition. A glimpse at (2) immediately reveals two unfavorable features: (i) the operator is badly nonlinear; (ii) ellipticity is lost at points where $\mathscr{D}u = 0$. The strong nonlinearity makes it impossible to develop a potential theory along the lines of the classical one. p-harmonic functions do not enjoy integral representation formulas such as (3); there is no Green function, or Kelvin transform, or Poisson kernel. p-subharmonicity is not preserved by the classical mollification processes, as is the case for subharmonic functions. This makes it impossible to regularize p-subharmonic functions. In retrospect, this obstruction is also deeply connected with (ii) above. The lack of ellipticity results in loss of regularity of p-harmonic functions. Through the fundamental work of several people, we know now that the optimal regularity of solutions to (2) is $C^{1,\alpha}$. Here is an easy example. The function $u(x) = |x|^{\frac{p}{p-1}}$ satisfies the equation $\Delta_p u = \text{const}$; but $u \notin C^2$, when $p > 2$.

Nonlinear potential theory of degenerate elliptic equations, by J. Heinonen, T. Kilpeläinen, and O. Martio, is an exceptionally well-thought-out, self-contained monograph on the potential theory (in a broad sense) of the so-called $\mathscr{A}$-harmonic operator

$$\operatorname{div} \mathscr{A}(x, \mathscr{D}u) = 0. \tag{9}$$

Here, $\mathscr{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping which satisfies the following assumptions:

(i) $x \mapsto \mathscr{A}(x, \xi)$ is measurable for every $\xi \in \mathbb{R}^n$, and $\xi \mapsto \mathscr{A}(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^n$.

(ii) There exist $1 < p < \infty$ and positive numbers α, β such that

$$\langle \mathscr{A}(x, \xi), \xi \rangle \geq \alpha \omega(x) |\xi|^p, \qquad |\mathscr{A}(x, \xi)| \leq \beta \omega(x) |\xi|^{p-1}$$

for every $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.

(iii) $\langle \mathscr{A}(x, \xi_1) - \mathscr{A}(x, \xi_2), \xi_1 - \xi_2 \rangle > 0$, whenever $\xi_1, \xi_1 \in \mathbb{R}^n$, and $\xi_1 \neq \xi_2$.

(iv) $\mathscr{A}(x, \lambda\xi) = \lambda|\lambda|^{p-2}\mathscr{A}(x, \xi)$ for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

The function ω in (ii) is assumed to be a p-admissible weight. This means that ω is a nonnegative measurable function on $\mathbb{R}^n$ such that the measure $d\mu = \omega dx$ satisfies on every ball $B \subset \mathbb{R}^n$:

(I) the doubling condition $\mu(2B) \leq C_I \mu(B)$;

(II) a Sobolev type embedding theorem

$$\left(⨍_B |u|^{kp}\, d\mu \right)^{\frac{1}{kp}} \leq C_{II} \operatorname{diam} B \left(⨍_B |\mathscr{D}u|^p \, d\mu \right)^{\frac{1}{p}},$$

whenever $u \in C_0^1(B)$, for some $k > 1$;

(III) a Poincaré type inequality

$$\int_B |u - u_B|^p \, d\mu \leq C_{III} (\operatorname{diam} B)^p \int_B |\mathscr{D}u|^p \, d\mu,$$

whenever $u \in C^1(B)$, with $u_B = ⨍_B u \, d\mu$;

(IV) a natural nondegeneracy condition which says that if $u_i \in C^\infty(\Omega)$, $\int_\Omega |u_i|^p d\mu \to 0$, and $\int_\Omega |\mathscr{D}u_i - v|^p d\mu \to 0$ as $i \to \infty$, then $v = 0$ a.e. in Ω. The model equation for (9) is the perturbed p-Laplacian $\operatorname{div}(\omega(x)|\mathscr{D}u|^{p-2}\mathscr{D}u)$. In this case, $\mathscr{A}(x, \mathscr{D}u) = \omega(x)|\mathscr{D}u|^{p-2}\mathscr{D}u$, and one can take $\alpha = \beta = 1$ in (ii) above. When $\omega \equiv 1$, one recovers the p-Laplacian introduced in (2). One of the main motivations for presenting matters in such generality comes from the theory of quasiregular mappings. It is not easy to explain this point without going into a deep investigation of the subject, but let us say a few words. Suppose that Ω, Ω' are open sets in $\mathbb{C}$ and that $f : \Omega \to \Omega'$ is holomorphic. Then $u \circ f : \Omega \to \mathbb{R}$ is harmonic, whenever $u : \Omega' \to \mathbb{R}$ is. Conversely, if the pullback of every harmonic function in Ω' is harmonic in Ω, then f is either holomorphic or antiholomorphic. Quasiregular mappings in $\mathbb{R}^n$ are the multidimensional analogue of holomorphic functions. A continuous mapping $f : \Omega \to \mathbb{R}^n$ is called *quasiregular* if $f \in [H^{1,n}_{\text{loc}}(\Omega)]^n$ and there exists $K \geq 1$ such that for a.e. $x \in \Omega$

$$|f'(x)|^n \leq K \, J_f(x).$$

Here, $f'(x)$ denotes the formal differential of f at x, whereas $J_f(x)$ is the determinant of the Jacobian of f at x. An important aspect of the theory is that the above-mentioned "pullback principle" has its counterpart for such mappings if one replaces harmonic functions with $\mathscr{A}$-harmonic ones; i.e., solutions of equation (9). For instance, it is an easy calculation to show that the function $\psi(x) = \log|x|$ is a solution to the equation (2) in $\mathbb{R}^n \setminus \{0\}$ when $p = n$. The following important result (see Theorem 14.19 in the book under review) illustrates the present situation.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ be a nonconstant K-quasiregular mapping, and $b \in \mathbb{R}^n$. Then the function $u(x) = \psi(f(x) - b) = \log|f(x) - b|$ is $\mathscr{A}$*-harmonic *in the open set $\Omega \setminus f^{-1}(b)$, with an $\mathscr{A}(x, \xi)$ satisfying* (i)–(iv) *above with $\omega \equiv 1$, $\alpha = K^{-1}$, and $\beta = K$.*

This theorem has notable topological implications when coupled with nonlinear potential theoretic results involving capacity and Hausdorff dimension.

One of them is that *nonconstant quasiregular mappings are discrete and open.*

We cite the authors: "Potential theoretic methods are of great importance in the study of quasiregular mappings, even more so than in classical complex function theory where other means are often available." The book by Heinonen, Kilpeläinen, and Martio represents veritable witness of the above statement. In sixteen chapters the reader is taken from the basic properties of Sobolev functions and variational capacities to the latest developments in the theory of quasiregular mappings, nonlinear potential theory, and the deeply interconnected theory of quasilinear partial differential equations modelled on (2). Each chapter ends with some interesting historical notes, in which references to the most recent literature can also be found. The bibliography is exhaustive and contains a wealth of material.

One of the many highlights of the book is the solution in Chapter 9 of the Dirichlet problem for equation (9) by adapting the Perron–Wiener–Brelot method discussed above. This adaptation is by no means straightforward and requires a great deal of preparatory work and new ideas. Just to give a flavor of the difficulties involved, we mention that $\mathscr{A}$-superharmonic functions are introduced and studied in Chapter 7. Following a standard approach in potential theory, part (iii) in the above-recalled Riesz's definition is replaced with the assumption:

(iii)$'$ For each open $D \Subset \Omega$ and each $h \in C(\overline{D})$ which solves the $\mathscr{A}$-harmonic equation (9) in D, the inequality $u \geq h$ on ∂D implies $u \geq h$ in D.

In order to carry Perron's method, one now needs to know that for a sufficiently large class of open sets one can actually solve the Dirichlet problem. For Laplace's equation the Poisson kernel provides an explicit solution for a ball, and balls constitute a basis for the topology of $\mathbb{R}^n$. This poses the problem of finding sufficient conditions for regularity of boundary points for the $\mathscr{A}$-harmonic equation (9). This program is carried in Chapter 6, where, among other important results, it is shown that the Wiener type condition (see (5) above)

$$\int_0^1 \left[\frac{\mathrm{cap}_{p,\mu}((\mathbb{R}^n \setminus \Omega) \cap B_r(x_0)\,;\, B_{2r}(x_0))}{\mathrm{cap}_{p,\mu}(B_r(x_0)\,;\, B_{2r}(x_0))}\right]^{\frac{1}{p-1}} \frac{dr}{r} = \infty \tag{10}$$

is sufficient for the continuity up to the boundary of the variational solution to the Dirichlet problem. In (10) $\mathrm{cap}_{p,\mu}$ denotes the variational p-capacity defined analogously to (6) but with respect to the measure $d\mu = \omega dx$. In particular, (10) shows that balls and polyhedra are regular for the Dirichlet problem (in fact, it shows that any domain Ω which in the language of [16] has an exterior "corkscrew" at every $x_0 \in \partial\Omega$ is regular). This result motivates the definition of Poisson modification of an $\mathscr{A}$-superharmonic function in Chapter 7. In Chapter 9 this tool is used in connection with the Harnack inequality established in Chapters 3 and 6 and a topological lemma of Choquet to prove the analogue of Perron's result. Also, a generalization of a result of Kellogg is given. The latter states that the set of irregular boundary points of an open set has weighted p-capacity zero. We also would like to mention another important achievement in Chapter 7; namely, that $\mathscr{A}$-superharmonic functions which are locally bounded are, in fact, weak supersolutions of equation (9). This rather intriguing and nontrivial result says, in particular, that (locally bounded) $\mathscr{A}$-superharmonic functions belong to the weighted Sobolev space $H^{1,p}_{\text{loc}}(\Omega; \mu)$. This fact enables one to employ variational methods in potential theoretic problems and vice-versa.

It should be remarked that only recently several important contributions have been made to various basic questions regarding the Dirichlet problem. For instance, the resolutivity of continuous functions (the analogue of Wiener's result recalled above) in the nonlinear case was proved by Lindqvist and Martio [21] in 1985 for $p = n$ and by Kilpeläinen [18] in 1989 for all $p > 1$ (see Theorem 9.25 in the book under review). It is a difficult open problem whether semicontinuous functions are resolutive (personal communication by P. Lindqvist). A recent fundamental contribution is the final settlement of Kilpeläinen and Malý [19] of the necessity of Wiener's condition (10) in the range $1 < p \leq n - 1$. In the unweighted case the sufficiency of (10) was established by Maz'ya [23] in 1970. Ironically, but perhaps not surprisingly, what in the linear theory is the easy part of Wiener's criterion, the necessity, has constituted in the nonlinear setting a long-standing difficult open problem. In 1985 Lindqvist and Martio solved the problem in the affirmative in the case $p > n - 1$, but their method could not be extended to the case $1 < p \leq n - 1$.

The last part of the book (Chapters 13–15) links nonlinear potential theory to the theory of quasiregular mappings. Chapter 14 is devoted to such mappings and represents the only exception to the self-contained character of the book. It could not have been otherwise, given the deep and technical nature of the results involved. On the other hand, the exposition is masterfully organized, and precise references are always provided. The important connection between quasiregular mappings and the $\mathscr{A}$-harmonic equation (9) is established in detail (see Theorem 14.39). A fundamental question in the subject is whether Picard-type results hold for entire quasiregular mappings. In 1967 Zorich [36] conjectured that omission of two points is not possible. In a deep paper in 1985 Rickman [27] disproved this conjecture, showing that in $\mathbb{R}^3$ any number of points can be omitted. In 1991 Eremenko and Lewis [11] found a remarkable and insightful approach to Rickman's Picard theorem which combines nonlinear potential theory with a priori estimates for solutions of (9). Their proof is presented in detail in Section 14.58. Chapter 15 is devoted to showing the admissibility of two important classes of weights: Muckenhoupt's A_p-weights and powers of the Jacobian of a quasiconformal mapping (this is a quasiregular mapping which is also a homeomorphism). A celebrated theorem of Gehring [14] states that if $f : \Omega \to \Omega'$ is quasiconformal, then its Jacobian J_f is an A_∞-weight of Muckenhoupt. Thereby, there exists $1 < q < \infty$ such that $J_f \in A_q$. In [12] it was shown that A_p-weights are p-admissible and that for every quasiconformal mapping f in $\mathbb{R}^n$, $J_f^{1-\frac{2}{n}}$ is 2-admissible. Using Gehring's result, in Theorem 15.33 it is proved that $J_f^{1-\frac{p}{n}}$ is p-admissible for every $1 < p < n$.

Two appendices, on existence of solutions to the $\mathscr{A}$-harmonic equation (9) and on the John–Nirenberg's theorem for doubling measures on $\mathbb{R}^n$, complete the book.

Some final comments. The study of potential theory requires a certain amount of effort. The book by Heinonen, Kilpeläinen, and Martio is no exception in this respect. If, however, one is willing to learn the subject and, at the same time, become deeply acquainted with its latest and exciting developments, then this is the place to start. We would like to add that the self-contained character of this book and the excellent organization of the material make it a perfect source for a graduate course. We are grateful to the authors for having undertaken the writing of this book.

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Nearrings—geneses and applications, by James R. Clay. Oxford University Press, London, 1992, x+469 pp., $95.00. ISBN 0-19-853398-5

A *nearring* (or near-ring) satisfies all axioms of an associative ring, except commutativity of addition and one of the two distributive laws. If the nearring N satisfies the left distributive law $a(b+c) = ab + ac$, then N is called a *left nearring.* A *right nearring* is, of course, a nearring satisfying the right distributive law. If the (left or right) nearring N satisfies the condition $On = nO = O$ for all $n \in N$, where O denotes the neutral element of the group $(N, +)$, then N is called *O-symmetric.* A nearring N with the property that $(N \backslash \{O\}, \cdot)$ is a group is called a *nearfield.* The additive group of a nearfield is always abelian. In this review, all nearrings under consideration will be left nearrings.

The first and fairly comprehensive treatise on nearrings by G. Pilz [P] appeared in 1977 and in revised form in 1983. The monograph by J. D. P. Meldrum, *Near-rings and their links with groups* [M], published in 1985, was intended as an introduction to the subject but also contains some deeper material on the group-theoretic aspect. The highly developed theory of nearfields is covered by Heinz Wähling's treatise *Theorie der Fastkörper* [W], which was