BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, January 1995

0273-0979/95 \$1.00 + \$.25 per page

Arithmetic of probability distributions and characterization problems on Abelian groups, by G. M. Fel'dman. Translations of Mathematical Monographs, vol. 116 (Simeon Ivanov, ed.), American Mathematical Society, Providence, RI, 1993, v + 223 pp., \$127.00. ISBN 0-8218-4593-4

Structural probability theory, as a topic of research that emphasizes the algebraic-topological properties of the ranges of random variables, has its roots in the pioneering memoirs of A. I. Khintchine and P. Lévy of 1937 and 1939, respectively. Already at that time the great masters of probability theory recognized the value of knowledge arising from studying convolutions of probability distributions, which in turn lead to more detailed investigations of the algebraic-topological nature of their domain of definition. While Khintchine emphasized decomposability in the sense of convolution within the classical framework of the real line, Lévy started a theory of probability on the torus $\mathbb{T}$. In the above-mentioned publications the authors disclose their visions of an arithmetic in the convolution semigroup $M^1(X)$ of all probability measures on a topological group X. The idea of enlarging upon arithmetic properties of probability measures on the traditional groups $\mathbb{R}$ and $\mathbb{T}$ was promoted by D. Dugué in his *Arithmétiques des lois de probabilités* [2] of 1957 and further elaborated five years later by Yu. V. Linnik in his book *Décomposition des lois de probabilités* [5]. In 1977 the monograph *Decomposition of random variables and vectors* by Yu. V. Linnik and I. V. Ostrovskii appeared, in which the euclidean groups $\mathbb{R}^d$ of arbitrary dimension $d \geq 1$ gained increasing importance.

Nowadays we visualize a wealth of literature on probability theory on topological groups (including topological vector spaces, homogeneous spaces, double coset spaces, and other structures arising from topological groups). The reviewer's monograph of 1977 (with expanded Russian translation of 1981), which is based on previous works of U. Grenander (1963) and K. R. Parthasarathy (1967), contains a thorough description of the central limit problem for measures on an arbitrary locally compact group. Although in this monograph some basic arithmetic of $M^1(X)$ is touched upon, the topic as such lies outside the intentions of the book.

In 1988 I. Z. Rusza and G. J. Székely presented their *Algebraic probability theory* as an approach to "probability theory without probabilities" (see the reviewer's essay on that book which appeared in the *Bulletin of the London Mathematical Society* **22** (1990), 523–524). In this slightly esoteric contribution to the research literature, decomposability in the semigroup $M^1(X)$ of probability measures on a topological group X is developed from decomposability in general semigroups. Naturally the results obtained are less specific or refined than those achieved for the groups $\mathbb{R}$ and $\mathbb{T}$. What the authors favorably exploit is D. G. Kendall's and R. Davidson's "Delphic theory" of 1968, and they are successful in coping with Khintchine's famous decomposition theorems. Exactly these profound results were the starting points of the extended investigations of Linnik and Ostrovskii and their schools. While R. Cuppens, inspired by E. Lukacs, devoted his book of 1975 to that part of arithmetic in $M^1(\mathbb{R}^d)$, the originators together with L. Z. Livshits and G. P. Chistyakov stressed the aspect of generalization to groups in various publications around 1975. It was shortly after 1975 when G. M. Fel′dman's first papers on decomposition problems for probability measures on a locally compact Abelian group appeared. Here was a student of Ostrowskii's who seemed to be predestined to enrich the work emanating from the Charkov school of structural probability theory. With remarkable energy Fel′dman pushed forward some of the decomposition and characterization problems to their natural limits in the sense that he achieved, within the class of all locally compact Abelian groups, a precise description of the domain of validity of the corresponding theorems.

The book under review reflects essentially the author's own work on the subject which, of course, includes most of the previous achievements in the theory.

We shall describe a selection of results from Chapters II and III that are typical for the approach and are ultimate with respect to attempts at further generalization.

To fix some notation: X will abbreviate a second countable locally compact Abelian group (written additively, with neutral element 0), and $M^1(X)$ will be furnished with the weak topology and with convolution $(\mu, \nu) \to \mu * \nu$ as its operation defined by

$$\mu * \nu(B) = \int_X \mu(B - x)\nu(dx)$$

for all Borel subsets B of X. Clearly, $M^1(X)$ becomes a topological semigroup with the Dirac measure ε_0 at 0 as unit element. Prominent subclasses of $M^1(X)$ are the classes $I(X)$ and $ID(X)$ of (weakly) idempotent and infinitely divisible measures in $M^1(X)$, respectively. Here the weakness added to the notions signifies admissibility of shifts (by Dirac measures) in the natural definitions. The elements of $I(X)$ are just the normed Haar measures of compact subgroups of X, and those of $ID(X)$ admit a canonical decomposition (à la Lévy-Khintchine) separating an idempotent, a Gaussian measure, and a generalized Poisson measure. The notion of a factor (or divisor) of a measure in $M^1(X)$ is straightforward. A nondegenerate (non-Dirac) measure in $M^1(X)$ is called indecomposable if it has only degenerate measures or shifts as factors.

Chapter II opens with the discussion of group analogs of the Khintchine decomposition theorems on the class $I_0(X)$ of all measures in $M^1(X)$ admitting neither indecomposable nor nondegenerate idempotent factors. The theorems proved by K. R. Parthasarathy, R. Ranga Rao, and S. R. S. Varadhan in 1963 read as follows:

(K1) Any $\mu \in M^1(X)$ admits a representation $\mu = m * \lambda_1 * \lambda_2$ where m is the maximal idempotent factor of μ, λ_1 is a (finite or countable) convolution of indecomposable measures, and $\lambda_2 \in I_0(X)$.

(K2) $I_0(X) \subset ID(X)$.

Gaussian distributions on X are introduced along the ideas of Parthasarathy et al., K. Urbanik, and S. N. Bernstein; they will be collected in the sets $\Gamma(X) :=$ $\Gamma_P(X)$, $\Gamma_U(X)$, and $\Gamma_B(X)$, respectively. Here a measure $\mu \in M^1(X)$ is said to be Gaussian (in the sense of Parthasarathy, i.e., an element of $\Gamma(X)$) if there exist an $x \in X$ and a nonnegative quadratic form ϕ on the dual Y of X such that the Fourier transform of μ admits a representation

$$\hat{\mu}(y) = (x, y)\exp(-\phi(y))$$

valid for all $y \in Y$. It turns out for embeddable measures $\mu \in M^1(X)$ in the sense that there exists a one-parameter convolution semigroup $(\mu_t : t \in \mathbb{R}_+)$ in $M^1(X)$ with $\mu_t \underset{\sim}{\to} \varepsilon_0$ as $t \underset{\sim}{\to} 0$ such that $\mu_1 = \mu$ that $\mu \in \Gamma(X)$ iff

$$\frac{1}{t}\mu_t(\complement U) \to 0$$

as $t \to 0$ for all Borel neighborhoods U of 0. The above mentioned classes of Gaussian distributions have been discussed in the reviewer's monograph of 1977. Fel′dman's completion of those results will be given subsequently. A central result of Chapter II is the description of the domain of validity of Cramér's decomposition of the classical Gaussian distribution. At first we note that H. Cramér showed in 1936 that all factors of a measure $\mu \in \Gamma(\mathbb{R}^d)$ belong to $\Gamma(\mathbb{R}^d)$. On the other hand, it has been known since the work of J. Marcinkiewicz (1939) that any measure in $\Gamma(\mathbb{T})$ has non-Gaussian factors. The ultimate solution to the generalized Cramér problem appears as Theorem 5.22: For a measure in $\Gamma(X)$ to admit only factors in $\Gamma(X)$ it is necessary and sufficient that X contains no subgroup topologically isomorphic to $\mathbb{T}$.

In a similar way the domain of validity of Raikov's theorem on the decomposition of Poisson measures in $M^1(X)$ has been characterized by A. L. Rukhin (1970) and G. M. Fel′dman (1977). The corresponding results on the set $P(X)$ of Poisson distributions of the form $\exp(a\varepsilon_{x_0})$ for $a > 0$ and $x_0 \in X$ are discussed in §6 of the present book. Another most remarkable achievement is the generalization to Abelian groups of Linnik's theorem on the factors of the convolution of a Gaussian and a Poisson measure in $M^1(\mathbb{R})$. The generalization due to the author and A. E. Fryntov (1981) appears as Theorem 7.2: Given a measure γ in the set $\Gamma^s(X)$ of symmetric Gaussian distributions on X such that $\operatorname{supp}\gamma$ is a connected finite-dimensional subgroup of X and a measure $\pi \in P(X)$, then $\gamma, \pi \in I_0(X)$ implies that $\gamma * \pi \in I_0(X)$.

From the general results on the measures in $I_0(X)$ we quote the ultimate statement concerning the richness of $I_0(X)$. In Theorem 8.7 it is stated that for any Abelian group X the set $I_0(X)$ is dense in $ID(X)$ iff X is either nondiscrete or $X \cong \mathbb{Z}(2)$. Applying a considerable amount of structure theory (of locally compact Abelian groups), Fel′dman adds to his book his description of those groups X for which the class $I_0(X)$ forms a "basis" within $ID(X)$. More precisely, in Theorem 8.8 he proves the equivalence of the following statements:

(i) Every measure in $ID(X)$ can be represented as a (finite or countable) convolution product of measures in $I_0(X)$.
(ii) $X \cong \mathbb{R}^d \times D$ where $d \geq 0$ and D denotes a discrete group without elements of (finite) order > 2.

Chapter III concerns probabilistic and statistical characterizations of Gaussian distributions on a (second countable locally compact Abelian) group X. A first characterization of the class $\Gamma(X)$ in terms of the class $\Gamma_U(X)$ was already given in Chapter II. Here we apply the definition $\gamma \in \Gamma_U(X)$ if $y(\gamma) \in \Gamma(\mathbb{T})$

for all $y \in Y$. Obviously, $\Gamma(X) \subset \Gamma_U(X)$. And by Theorem 5.36 the equality $\Gamma(X) = \Gamma_U(X)$ holds iff the factor group of Y by the subgroup of all compact elements of X is topologically isomorphic to $\mathbb{Z}$. For the comparison of $\Gamma(X)$ with the class $\Gamma_B(X)$ we recall that by definition $\mu \in \Gamma_B(X)$ if for any pair ξ, η of independent X-valued random variables with common distribution μ, the random variables $\xi + \eta$ and $\xi - \eta$ are also independent. It is easy to see that $\mu \in \Gamma_B(X)$ iff

$$\hat{\mu}(y_1 + y_2)\hat{\mu}(y_1 - y_2) = \hat{\mu}(y_1)^2|\hat{\mu}(y_2)|^2$$

for all $y_1, y_2 \in Y$. In view of the comparison of the classes $\Gamma(X)$ and $\Gamma_B(X)$ we first note that $\Gamma(X) \subset \Gamma_B(X)$ but also observe that $\Gamma_B(X)$ may contain idempotents which we collect in the set $I_B(X)$. Theorem 9.10, based on the work of L. Corwin (1970) and the reviewer together with C. Rall (1972), states that

$$I_B(X) * \Gamma(X) = \Gamma_B(X)$$

iff for any compact Corwin subgroup K of X the factor group X/K does not contain a subgroup $\cong \mathbb{T}^2$. Here K is called a Corwin group provided $K = K^{(2)} := f_2(K)$ under the mapping $f_2 \colon x \to 2x$ on K. The final comparison then appears as Proposition 9.13:

$$\Gamma(X) = \Gamma_B(X)$$

iff the only compact Corwin subgroup of X is $\{0\}$.

In the remaining part of the chapter the author's aim is to characterize those groups for which statements analogous to the classical Darmois-Skitovich (around 1954) and Polya-Linnik (1923, 1954) theorems are true. A sequence $(a_j \colon 0 \le j \le m)$ of integers is said to be admissible for the given group X if $X^{(a_j)} \ne \{0\}$ for all $j = 1, \dots, m$. Then the first-mentioned characterization is contained in Theorem 10.13: Let $\xi_1, \dots, \xi_m$, $m \ge 2$, denote independent random variables ξ_j with distributions $\mu_j \in M^1(X)$ such that

$$\prod_{j=1}^{m} \hat{\mu}_j(y) \ne 0$$

for all $y \in Y$. Moreover, let $(a_j \colon 0 \le j \le m)$ and $(b_j \colon 0 \le j \le m)$ be two admissible sequences for X. The following statements are equivalent:

(i) The independence of the linear forms $a_1\xi_1 + \cdots + a_m\xi_m$ and $b_1\xi_1 + \cdots + b_m\xi_m$ implies $\mu_j \in \Gamma(X)$ for all $j = 1, \dots, m$.

(ii) X is torsion free or $X^{(p)} = \{0\}$ for some prime number p.

For the second-mentioned characterization the class $\Gamma_A(X)$ where $A := (a_j \colon 0 \le j \le m)$ is introduced as the class of measures $\mu \in M^1(X)$ enjoying the following property: If $\xi_1, \dots, \xi_m$, $m \ge 2$, are independent identically distributed X-random variables with distribution $\mu \in M^1(X)$, then the linear forms $a_0\xi_1$ and $a_1\xi_1 + \cdots + a_m\xi_m$ are identically distributed. One notes that $\mu \in \Gamma_A(X)$ if $\hat{\mu}$ satisfies the functional equation

$$\hat{\mu}(a_0 y) = \prod_{j=1}^{m} \hat{\mu}(a_j y)$$

valid for all $y \in Y$. Let $\mathscr{A}(X)$ abbreviate the collection of all admissible sequences $A = (a_j : 0 \leq j \leq m)$, $m \geq 2$, where the integers a_j are relatively prime and satisfy the equality

$$a_0^2 = a_1^2 + \cdots + a_m^2.$$

For $A \in \mathscr{A}(X)$ this equality implies that $\Gamma^s(X) \subset \Gamma_A(X)$. Since $\Gamma_A(X)$ is a semigroup, one therefore has

$$I_A(X) * \Gamma^s(X) \subset \Gamma_A(X)$$

where $I_A(X)$ denotes the set of idempotents in $\Gamma_A(X)$. Now, as in the case of measures in $\Gamma_B(X)$, the author proves the characterization Theorem 11.9.:

$$I_A(X) * \Gamma^s(X) = \Gamma_A(X)$$

for every $A \in \mathscr{A}(X)$ iff $X \cong \mathbb{R}^d \times D$ where $d \geq 0$ and D is a discrete torsion-free group or iff $X^{(p)} = \{0\}$ for some prime number p.

The monograph under review is a highly specialized piece of work. The topics treated are chosen from the author's research projects, on which he has worked since 1977. The expertise demonstrated in the technical development of the proofs relies on a skillful handling of the structure theory of locally compact Abelian groups and of the theory of analytic functions entering in a refined application of Fourier transforms of probability measures on the groups $\mathbb{R}^d$ and $\mathbb{T}^d$. Within the arithmetic of $M_1(X)$ (Chapter II) the author's main goal is the study of the structure of the class $I_0(X)$. As for the characterization of the Gaussian distributions on X (Chapter III) he concentrates on the generalization to groups of the famous classical Bernstein, Darmois-Skitovich, and Polya-Linnik theorems involving independence and identical distribution of linear forms over the integers. Chapter I on auxiliary results from the harmonic analysis on locally compact Abelian groups and from the theory of analytic functions and Appendices 1–4 on further topics supplement the main body of the monograph. From the comments in §§4–11 and from the appendices the reader can take precise references. With regret the reviewer misses the author's characterization of the Cauchy distribution on a group; the result submitted for publication in 1989 would have added favorably to Chapter III since the tools and preparations would have already been contained in the book.

The systematics of the exposition have been dictated by the special choice of topics. Specialists working in the field of decomposition theory for probability measures on groups will appreciate the restrictions to the absolutely necessary. The present English version of Fel′dman's book suffers from the pardonable insufficiencies common to translations performed by people for whom Russian is their mother tongue. The translation editor obviously agreed with the slightly unsatisfactory result. The number of mathematical inaccuracies is small; readers also will not be confused by too many (obvious) misprints.

There is no question that it was a wise decision by the editors of the *Mathematical Monographs Translations* series to accept this piece of work for publication and make it accessible to a broad community of mathematicians working in structural probability theory. Moreover, the handy monograph of a few more than 200 pages advertises a most interesting aspect of probability theory to all analysts who want to see abstract harmonic analysis at work. A sequence of fifteen unsolved problems added to the fourth appendix may serve as an initiation for future contributors to the progress of the theory.

References

[1] R. Cuppens, *Decomposition of multivariate probability*, Academic Press, New York, San Francisco, and London, 1975.

[2] D. Dugué, *Arithmétique des lois de probabilités*, Mém. Sci. Math., Fasc. 137, Gauthier-Villars, Paris, 1957.

[3] G. M. Fel′dman, *Cauchy distribution on abelian groups and its characterization*, Theory Probab. Appl. **36** (1991), 670–681.

[4] A. I. Khintchine, *Contribution à l'arithmétique des lois de distribution*, Bull. Univ. d'Etat Moscou Sér. Internat. Sect. A Math. Méch. **1** (1937), 6–17.

[5] Yu. V. Linnik, *Décomposition des lois de probabilitès*, Gauthier-Villars, Paris, 1962.

Herbert Heyer
Mathematisches Institut, Eberhard-Karls-Universitat Tübingen

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, January 1995

0273-0979/95 $1.00 + $.25 per page

Hopf algebras and their actions on rings, by S. Montgomery. CBMS Regional Conference Series in Mathematics, vol. 82, American Mathematical Society, Providence, RI, 1993, xiv + 238 pp., $32.00. ISBN 0-8218-0738-2

This book is a good guidebook to recent developments in Hopf algebra theory with an emphasis on their actions and coactions on algebras. It contains some classical results which overlap with the foregoing textbooks [S] and [A] but includes new applications and treatments. Most of the results have not previously appeared in book form. They are described in a self-contained way with or without proofs; and when the proofs are omitted, appropriate references to the literature are specified. This book could also be used as a textbook for graduate students.

Kaplansky [K] proposed ten conjectures on Hopf algebras in 1975. Some of them have recently been solved, including a kind of Lagrange's theorem for Hopf algebras settled by Nichols and Zoeller in 1989, which tells that a finite-dimensional Hopf algebra is free over any Hopf subalgebra. Chapter 3 is devoted to its proof, applications, and related topics. The authors chose a good time to publish such a basic theorem in book form, for it is useful and sometimes indispensable in the subsequent discussion of finite-dimensional Hopf algebras. Other contributions to Kaplansky's conjectures are mentioned without proof (Theorems 2.5.2, 2.5.3 and Conjecture 3.2.2).

The author, S. Montgomery, has contributed much to group actions on rings [M] and their generalizations to Hopf algebras (especially finite dimensional). One purpose of this monograph seems to develop the author's and her collaborators' results systematically, including prerequisites, backgrounds, and other related topics.

There have been many studies on group actions on rings and their relations with Galois theory [P]. When a Hopf algebra H acts on an algebra A, one can form the subalgebra of invariants A^H and the smash product $A\#H$, a generalization of the skew group ring, which is further generalized to the crossed product $A\#_\sigma H$ involving a two-cocycle σ. Ring-theoretic studies of the algebras A^H, $A\#H$, and $A\#_\sigma H$ as well as relations with Hopf Galois theory are