BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, January 1995

0273-0979/95 $1.00 + $.25 per page

Introduction to functional differential equations, by Jack K. Hale and Sjoerd M. Verduyn Lunel. Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993, x + 447 pp., $49.00. ISBN 0-387-94706-6

I

Functional differential equations (FDEs) are differential equations for functions x of a scalar variable which contain terms with x and x' evaluated at different arguments. The number of FDEs used in the sciences and in applied areas has been rapidly increasing throughout the past three decades. Applications range from enzyme kinetics and physiology to foodwebs and whaling control, from electronic circuit design to laser optics, and from studies of car traffic to the theory of business cycles.

One of the best-known examples is the equation

$$x'(t) = -\alpha x(t-1)[1 + x(t)] \tag{1}$$

with $\alpha > 0$, which was studied in a beautiful paper by Wright [2][1]. He ascribed his interest in equation (1) to the fact that it occurred in a heuristic proof of the prime number theorem (Wright [1]). Equation (1) is equivalent to the differential equation for logistic growth with a delayed argument in the affine linear factor. The latter had been discussed by Hutchinson in connection with oscillating population densities in constant environments [H]. For a nice study of equation (1) in this form, see [KM].

Setting $x = \log(1+y)$ for $y > -1$, we are led to the equation

$$x'(t) = f(x(t-1)) \tag{2}$$

with $f(\xi) = \alpha(1 - e^{\xi})$. Equation (2) with an arbitrary continuous function $f: \mathbb{R} \to \mathbb{R}$ is the simplest differential equation for a system governed by delayed feedback.

Equation (2) is easily solved if initial values $\phi(t)$ are prescribed on an interval whose length equals or exceeds the delay $r = 1$: For

$$\phi \in C = C([-1, 0], \mathbb{R})$$

a repeated application of the integrated version of equation (2),

$$x(t) = x(n) + \int_{n-1}^{t-1} f(x(s))\, ds \quad \text{for integers } n \geq 0,\ t \in [n, n+1],$$

yields a uniquely determined continuous function $x: [-1, \infty) \to \mathbb{R}$ which coincides with ϕ on the initial interval $[-1, 0]$ and satisfies equation (2) for all $t > 0$.

Solutions $t \mapsto x_t$ with values in the space of the initial data are obtained by setting

$$x_t(s) = x(t+s) \quad \text{for } t \geq 0 \text{ and } s \in [-1, 0].$$

[1] Numerical citations are taken from the book reviewed, and lettered citations are taken from the references added below.

The introduction of segments x_t for $\mathbb{R}^n$-valued functions x of a scalar variable permits an explanation of what an autonomous retarded FDE is, namely, an equation of the form

$$x'(t) = F(x_t) \tag{3}$$

where F is a continuous map from an open subset of a Banach space

$$C([-r, 0], \mathbb{R}^n), \qquad r > 0,$$

into $\mathbb{R}^n$. Observe that equations (1) and (2) have the form (3). Existence, uniqueness, and continuous dependence of solutions $x\colon [-r, t_+) \to \mathbb{R}^n$, $t_+ > 0$, for the forward initial value problem associated with equation (3) are not difficult under reasonable smoothness assumptions on the functional F. However, in general there are no backward solutions, and phase curves $t \mapsto x_t$ may merge into each other in finite time.

The relations

$$\Phi(t, \phi) = x_t, \qquad t \geq 0, \qquad x_0 = \phi$$

constitute a semiflow on the phase space $C([-r, 0], \mathbb{R}^n)$ with certain smoothing and compactness properties, and the aim of the theory is to understand the organization of the phase space in terms of lower-dimensional separating invariant manifolds or sets.

FDEs share some properties with ordinary differential equations (ODEs), others with classes of partial differential equations (PDEs). Linear autonomous retarded FDEs define strongly continuous semigroups on the phase space which are not analytic (in contrast to the case of, say, parabolic PDEs). The spectra of their generators consist of isolated eigenvalues with finite multiplicities and are given by transcendental characteristic equations. The latter are obtained from the familiar Ansatz with exponential solutions. For example, $x\colon t \mapsto e^{\lambda t}$ is a solution of the equation

$$x'(t) = -\alpha x(t-1) \tag{4}$$

if and only if

$$\lambda + \alpha e^{-\lambda} = 0.$$

Only a finite number of eigenvalues may have a nonnegative real part. Consequently, center and unstable manifolds of equilibria are finite dimensional. Systems of eigenvectors of the generator are in general not complete, a fact linked to the existence of so-called small solutions which decay to 0 as $t \to \infty$ faster than any exponential.

II

The geometric theory of FDEs begins, as for other initial value problems, with linear autonomous equations, the variation-of-constants formula, and local invariant manifolds at stationary points. The following remarks highlight a few more advanced topics which may help to give an impression of the field within the whole dynamical systems area. We concentrate on autonomous retarded FDEs.

For linear equations the relation between small solutions and completeness of eigenvectors is an important problem which requires tools from complex analysis, Laplace transform theory, and distributions.

The choice of a phase space of continuous functions, which seems so natural when existence, uniqueness, and continuous dependence are considered, becomes problematic when the theory arrives at the variation-of-constants formula. The integral representation for phase curves requires the application of the semigroup to discontinuous functions outside the original phase space. Diekmann et al. (see Clément et al. [1]) developed the sun-star calculus, a machinery of dual spaces and semigroups which provides suitably enlarged phase spaces in an abstract setting.

Scalar equations may have oscillating and periodic solutions, in contrast to the case of ODEs. For example, $x: t \mapsto \cos(\frac{\pi}{2}t)$ is a solution to equation (4) with $\alpha = \frac{\pi}{2}$. The question of the existence of periodic solutions for nonlinear autonomous equations has attracted much attention since the early sixties. The first results of Jones, Grafton, Chow, Pesin, notably Nussbaum [3, 5], and others were linked with progress in fixed-point and bifurcation theory and, in particular, with the concept of ejectivity due to Browder [2]. Ejectivity is a weak instability property of fixed points modelled after a result of Wright [2] on solutions of equation (1) not tending to 0 as $t \to \infty$. Nowadays there are also other techniques for existence proofs, including the Poincaré-Bendixson theory ("ω-limit sets of bounded solutions are periodic orbits or heteroclinic chains connecting stationary points") for certain classes of equations [MS, S, W2]. Results on analyticity, on the range of periods, and on the shape of periodic solutions were obtained. The local and global Hopf bifurcation theory was developed; the Fuller index was extended. Presently the behaviour of phase curves close to periodic orbits is studied; there are results on the distribution of Floquet multipliers (see also [D]), on uniqueness [N] and stability, on hyperbolicity, and on global unstable manifolds of periodic orbits.

Mallet-Paret [1] proved a partial Kupka-Smale theorem which states that equations with all stationary points and periodic orbits hyperbolic are generic. The remaining problem concerning the transversal intersection of stable and unstable manifolds is still unsolved.

The structure of global attractors is being explored. Gradient flows on these attractors are not as typical as on the attractors for those PDEs which could be investigated. Instead there exist more general Morse decompositions which allow periodic and more complicated motion (Mallet-Paret [5], McCord and Mischaikow [1]). In case of equation (2) with a C^1-function f satisfying

$$f(0) = 0 \qquad \text{and} \qquad f'(\xi) < 0 \quad \text{for all } \xi \in \mathbb{R}$$

and a one-sided boundedness condition (as in the example given by Wright's equation (1)), the global attractor in a large positively invariant subset $\overline{S} \subset C$ is either the singleton $\{0\}$ or a 2-dimensional disk in C bordered by a periodic orbit [W2]; both cases occur.

Homoclinic and heteroclinic orbits connecting equilibria and periodic orbits have been obtained, and the existence of chaotic motion could be proved ([W1] and others). The problem of describing such chaotic motion in terms of symbolic dynamics led to the generalization of inclination lemmas (Hale and Lin [2]), hyperbolic sets, and the shadowing lemma from the case of diffeomorphisms in finite dimensions to arbitrary C^1-maps in Banach spaces [SW,L].

The relationship with ODEs on one side (the case of small delay) and with difference equations on the other is being investigated in work on singular perturbations.

Applications suggest the study of equations of the form

$$x'(t) = f(x(t), x(t - r(x(t)))),$$

i.e., with state-dependent delay. Recent work of Mallet-Paret and Nussbaum [5, 6] shows that other phase spaces must be considered here. New techniques are necessary, and it seems that phenomena unknown in the case of constant delays occur.

Other keywords to characterize the field of FDEs are Lyapunov functionals and Razumikhin theory, equations with unbounded delay, oscillation theory, partial functional differential equations, and neutral FDEs. The latter are more general than retarded equations and include, for example, equations of the form

$$x'(t) - cx'(t-1) = g(x(t), x(t-1))$$

with $c \neq 0$. The behaviour of the semigroups defined by autonomous linear neutral FDEs is in general quite different from the behaviour in case of retarded equations.

III

In 1977 Jack Hale's book *Theory of functional differential equations* appeared. It represented the state that the mathematical theory had reached and became a standard source and reference for work on FDEs. Now Sjoerd Verduyn Lunel and Jack Hale present a new version called *Introduction to functional differential equations*, which takes into account what has been achieved meanwhile. Some of these developments also affected basic parts of the theory. Accordingly the new book is not just a second edition with material added. More than two thirds of the old book have been changed considerably. This refers in particular to the abstract chapter on dissipative systems and attractors, the improved functional analytic presentation of linear retarded FDEs and their adjoints, and the theory for neutral equations which is now much closer to the theory for the retarded ones. Results on characteristic matrices (Kaashoek and Verduyn Lunel [1]) and characteristic equations on small solutions and completeness of eigenvectors have been added. Center manifolds have been included. Smoothness of local invariant manifolds is proved in a new manner.

Every chapter ends with a section "Supplementary remarks" on results, methods, and problems beyond the scope of the book. Chapter 12 describes directions into which the field is developing or where interesting questions wait for answers. Topics chosen are FDEs on manifolds, the Kupka-Smale theory, attractors and Morse decompositions, singular perturbations and the relation to difference equations, averaging, and equations with unbounded delays.

Not much changed were the existence theory in Chapter 2, which concentrates on the $\mathbb{R}^n$-valued solutions and does not say much about smoothness of the semiflow, the chapter on the Lyapunov stability theory, and the material in Chapter 11 on the existence of periodic solutions of autonomous retarded equations. For equations not necessarily close to linear ones this existence theory is based on ejectivity, which leads to theorems with weak hypotheses but does not completely reflect the behaviour of phase curves close to unstable equilibria. It is not discussed that the unstable behaviour is in fact better than indicated by ejectivity and permits finding periodic solutions rather easily by Schauder's theorem or by an elementary index calculation. The proof of the main nonlocal result on the existence of periodic solutions, Theorem 2.3 in Chapter 11, contains the same unexplained passage as in the old version (p. 338, lines 7 and 8 on the growth of the Lyapunov functional along phase curves outside the set K; see Alt [1] and Walther [5] for other approaches using the same Lyapunov functional).

What is not contained but would have been desirable? A theorem in Chapter 8 on the relationship between Floquet multipliers of periodic orbits and spectra of linearized Poincaré maps might be mentioned—since this is used later on and, to my knowledge, is not available in the literature for the semiflows considered. Further, one could think of supplementary sections on local invariant manifolds for smooth, not necessarily invertible, maps in Banach spaces.

The book is written in a lively style which makes the reader feel the engagement of the authors in their subject. They emphasize motivation, concepts, and techniques of proof and discuss a variety of interesting open problems. Proofs of more advanced results are not always carried out in detail. The number of minor inconsistencies and typographical errors is relatively large; apparently the publisher spared a careful reading of the preprint.

It remains to underline that *Introduction to functional differential equations* is a book which is indispensable for researchers in this field. No other book on FDEs is as comprehensive. It covers many, if not most, of the important issues, keeps the field together, and may well serve as a stimulating guide into future research.

References

[D] P. Dormayer, *An attractivity region for characteristic multipliers of special symmetric solutions of $\dot{x}(t) = \alpha f(x(t-1))$*, J. Math. Anal. Appl. **168** (1992), 70–91.

[H] G. E. Hutchinson, *Circular causal systems in ecology,* Ann. New York Acad. Sci., vol. 50, New York Acad. Sci., New York, 1948, pp. 221–246.

[KM] S. Kakutani and L. Markus, *On the nonlinear difference-differential equation $y'(t) = [A - By(t-\tau)]y(t)$*, Contributions to the Theory of Nonlinear Oscillations IV, Princeton Univ. Press, Princeton, NJ, 1958.

[L] B. Lani-Wayda, *Hyperbolic sets, shadowing and persistence for noninvertible mappings in Banach spaces,* preprint, Univ. München, 1992.

[MS] J. Mallet-Paret and G. Sell, *Systems of delay differential equations with discrete Lyapunov functions,* preprint, Brown Univ., Providence, RI, 1993.

[N] R. D. Nussbaum, *Uniqueness and nonuniqueness for periodic solutions of $x'(t) = -g(x(t-1))$*, J. Differential Equations **34** (1979), 25–54.

[S] R. A. Smith, *Poincaré-Bendixson theory for certain retarded functional differential equations,* Differential Integral Equations **5** (1992), 213–240.

[SW] H. Steinlein and H. O. Walther, *Hyperbolic sets, transversal homoclinic trajectories, and symbolic dynamics for C^1-maps in Banach spaces,* J. Dynamics Differential Equations **2** (1992), 325–365.

[W1] H. O. Walther, *Hyperbolic periodic solutions, heteroclinic connections and transversal homoclinic points in autonomous differential delay equations,* Mem. Amer. Math. Soc., vol. 79, no. 402, Amer. Math. Soc., Providence, RI, 1989.

[W2] ———, *The 2-dimensional attractor of $x'(t) = -(\mu)x(t) + f(x(t-1))$*, Mem. Amer. Math. Soc., vol. 113, no. 544, Amer. Math. Soc., Providence, RI, 1995 (to appear).

Hans-Otto Walther
Universität Giessen
E-mail address: hans-otto.walther@math.uni-giessen.de