BOOK REVIEW

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Introduction to regularity theory for nonlinear elliptic systems, by Mariano Giaquinta. Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1993, viii + 130 pp., \$29.00. ISBN 3-7643-2879-7

True mathematical understanding of nature is impossible without an understanding of the partial differential equations and variational principles that govern a large part of physics. Already very early in the development of calculus, besides the linear equations of electrostatics, for example, also nonlinear partial differential equations were studied. A prominent example is the nonparametric minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)=0\,,\qquad u\colon\Omega\subset\mathbb{R}^2\to\mathbb{R}\,,$$

which, in 1762, Lagrange derived as an illustration of what later became the Euler-Lagrange variational principle. The expression

$$\operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$$

(up to a factor) gives the mean curvature of the hypersurface $S\subset\mathbb{R}^3$ defined by the graph of u.

Looking back, it is no surprise that nonlinear partial differential equations first arose from an interplay of physics and geometry. In the eighteenth century, however, such a distinction would have been meaningless, as mathematics and physics were still largely being considered as a whole. Today many mathematicians and theoretical physicists are turning back to this view, largely because more and more examples emerge of nonlinear partial differential equations that play a fundamental role both in geometry and in physics: Harmonic maps, Yang-Mills equations, Einstein equations, etc.

Very often such equations arise from minimization problems

$$E(u)=\int_\Omega f(x\,,\,u(x)\,,\,\nabla u(x))\,dx\to\min\,,\tag{0.1}$$

where admissible comparison functions $u=(u^1\,,\,\dots\,,\,u^N)\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$ may be constrained, for instance, by boundary conditions, and where the function $f\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^{n\cdot N}\to\mathbb{R}$ is smooth in all its variables, bounded from below, and convex in the third variable. In this case, formally at least, problem (0.1) gives rise to an elliptic system of partial differential equations

$$-\frac{\partial}{\partial x^\alpha} f_{p^i_\alpha}(x\,,\, u(x)\,,\, \nabla u(x)) + f_{u^i}(x\,,\, u(x)\,,\, \nabla u(x)) = 0\,, \qquad 1 \le i \le N. \tag{0.2}$$

Here, $x = (x^1\,, \ldots\,, x^n)$, subscripts denote partial derivatives, and repeated Greek or Latin indices are summed from 1 to n or 1 to N , respectively. Typical examples are

$$f(x\,,\, u\,,\, p) = \tfrac{1}{2}|p|^2$$

or

$$f(x\,,\, u\,,\, p) = g_{ij}(u)\gamma^{\alpha\beta}(x)p^i_\alpha p^j_\beta\,,$$

where $\gamma = (\gamma_{\alpha\beta})_{1\le\alpha\,,\,\beta\le n}$ with inverse $\gamma^{-1} = (\gamma^{\alpha\beta})_{1\le\alpha\,,\,\beta\le n}$ and $g = (g_{ij})_{1\le i\,,\,j\le N}$ are uniformly positively definite matrices. In the first case, (0.1) then gives the Dirichlet integral

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx\,,$$

and (0.2) reduces to Laplace's equation

$$\Delta u = 0. \tag{0.3}$$

In the second case, (0.1) gives the energy of maps from a domain in $\mathbb{R}^n$, viewed as a coordinate patch on a manifold $\mathcal{M}$ with Riemannian metric γ , into $\mathbb{R}^N$ with metric g . If we also interpret the range as a coordinate chart on an N-dimensional manifold $\mathcal{N}$, (0.1) in this case yields a generalization of Dirichlet's integral to maps $u\colon \mathcal{M} \to \mathcal{N}$, and (0.2) becomes the equation

$$-\Delta_{\mathcal{M}} u^i = \gamma^{\alpha\beta}\Gamma^i_{ji}(u)u^j_{x^\alpha}u^k_{x^\beta} \tag{0.4}$$

for harmonic maps from $\mathcal{M}$ into $\mathcal{N}$, where Γ^k_{jk} denote the Christoffel symbols on $\mathcal{N}$.

Equation (0.4) is a model case for many problems of similar structure. In the same way that Laplace's equation (0.3) inspired potential theory (and eventually the analysis of linear partial differential equations, spectral theory, and functional analysis), nonlinear elliptic systems like (0.4) have sparked mathematical developments of great depth and fundamental importance.

In the 1960s a first systematic approach to nonlinear elliptic equations and systems was undertaken by Ladyženskaya and Solonnikov, who were able to profit from basic advances in linear partial differential equations with only measurable coefficients, due to De Giorgi, Nash, and Moser in the late 1950s. These techniques were quite sophisticated and unwieldy at first, and some efforts were made at improving them and obtaining sharp regularity and existence results of greater generality.

Giaquinta's book gives a state-of-the-art account of a large and important part of this field, as viewed by a leading expert. The book is focused around equations of type (0.4), characterized by a linear diagonal principal part and lower-order terms that depend quadratically on the gradient of u . First, the lower semicontinuity results (that yield existence of weak solutions to (0.4) by solving the minimization problem (0.1) in suitable Sobolev spaces) are explained in detail, including an excellent discussion of developments not treated in the book. Once existence of weak solutions is obtained, the next step is to improve their regularity and, ideally, to show that the weak solutions in fact are smooth and satisfy (0.2) classically. As a first method in this regard, the L^2-theory, or Hilbert space method, is presented, yielding smoothness of solutions of linear equations with smooth coefficients. More refined estimates are available from Schauder theory. In Giaquinta's book, this theory is developed based on ideas of Morrey and on Campanato's perturbation method—that is, free of potential theory—yielding Hölder continuity of a solution (and its first derivatives) to a linear elliptic partial differential equation

$$-\frac{\partial}{\partial x^\alpha}\left(a^{\alpha\beta}(x)\frac{\partial}{\partial x^\beta}u\right) = \frac{\partial}{\partial x^\alpha}f^\alpha$$

with Hölder continuous coefficients $a = (a^{\alpha\beta})$ and right-hand side $f = (f^\alpha)$. In the fourth chapter the L^p-estimates are presented, again avoiding potential theory but being based on an interpolation theorem of Stampacchia and the results of the two preceding chapters instead. Chapter 5 contains some beautiful new material related to the regularity theorem of De Giorgio-Nash and Moser's Harnack inequality; in particular, with the aid of a covering lemma due to Krylow-Safonov and some results of Di Benedetto-Trudinger, it is shown that Harnack's inequality holds for functions in the De Giorgi class. Finally, these ideas are applied to obtain partial regularity of minimizers of functionals of type (0.1), in particular, partial regularity of energy-minimizing harmonic maps.

The book summarizes the contents of a series of lectures given by Giaquinta in the Nachdiplom graduate program of the ETH Zürich in 1983-84. It is similar to notes by Giaquinta, published by Princeton University in 1983; however, it reaches beyond the latter in many important respects, for instance, regarding the regularity results for functions in De Giorgi's class mentioned above. The Birkhäuser series "Lectures in Mathematics ETH Zürich" was recently created as a means to make the notes of the ETH "Nachdiplom" lectures available to the mathematical community.

Giaquinta is an excellent expositor. Moreover, he has mainly contributed to the developments outlined in his notes. Anyone interested in partial differential equations would want to have this book on the shelf.

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