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P. Beazley Cohen and F. Hirzebruch
Collège de France
E-mail address, P. Beazley Cohen: pcohen@ccr.jussieu.fr
Max-Planck Inst. für Mathematik
E-mail address, F. Hirzebruch: hirz@mpim-bonn.mpg.de

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Banach and locally convex algebras, by A. Ya Helemskii. Clarendon Press, Oxford, 1993, xv + 446 pp., $90.00. ISBN 0-19-853578-3

It will be convenient to have before us an indication of the main topics covered in the book. The following outline of the contents contains only chapter and section headings but should adequately suggest the style and organization of the included material. The complete contents contains in addition the titles of numerous subparagraphs of the various sections.

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Chapter 0. Foundations

§0. Reminders from set theory, linear algebra, and topology
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Chapter I. Initial concepts and first results

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§3. Group algebras

This book, which is a translation from the Russian, is organized as a textbook. The textbook format is reflected in the numerous exercises, hints, challenging remarks, and many cross-references. Presumably it also accounts for the preliminary Chapter 0, which contains "the indispensable background material" needed in the text. The chapter covers, in addition to numerous expected topics for support of a general algebra-analysis book, a variety of special concepts plus the machinery and terminology needed for dealing with them.

The general style of the book, which is unorthodox and sometimes rather confusing, is illustrated by the following definition of an "abstract algebra" in §1 of Chapter I:

> Let A be a linear space. A bi-operator, $m: A \times A \to A$ is called a *multiplication bi-operator* or simply a *multiplication* in A if, writing ab instead of $m(a, b)$, it satisfies the identity $(ab)c = a(bc)$, $a, b, c \in A$. The space itself when furnished with a multiplication (in other words, the pair (A, m) consisting of a linear space and a multiplication given in it) is called a *complex associative algebra* or, more briefly, an *algebra*.

(Note that "bi-operator" is defined in the discussion of tensor products in Chapter 0.)

This is a trivial example, but it illustrates a more or less consistent style used throughout the book in dealing with a variety of technical concepts. These include, for example, the notions of modules, tensor products, and homology theory, plus the machinery needed to work with them. In addition, the author frequently uses the language and general approach of category theory. All of these things have at least an indirect influence on much of the material studied in the book and often actually dominate the discussion, especially in the last two chapters. It is accordingly very difficult to state in a reasonable amount of space most of the definitions and theorems in the book. The problem is usually the special technical character of the notation and concepts that need to be explained.

Despite these problems, it is perhaps worthwhile to describe very briefly the nature of one of these notions. A good example is the notion of a "module over an algebra A". Thus, let X be a linear space along with a "bi-operator"

$$m\colon AxX \to X\,,\ (a\,,\,x) \mapsto a\cdot x$$

such that, for each a in A, the mapping

$$m_a\colon X \to X\,,\ x \mapsto a\cdot x$$

is linear in x. If also

$$m_{ab} = m_a m_b\,,$$

for all a and b in A, then m is called a *left outer multiplication* of A on X and X is called a *left A-module.* It is obvious that the mapping

$$T\colon A \to \mathscr{L}(X)\,,\ a \mapsto m_a$$

is a homomorphism of A into the algebra of linear transformations on X and thus is a representation of A on X in the usual sense. Similarly, every such representation of A defines a left A-module. If both X and Y are A-modules, then a map $\varphi\colon X \to Y$ is called a *morphism of A-modules* if it is a linear operator and $\varphi(a\cdot x) = a\cdot\varphi(x)$, for all a and x. Modules over a fixed algebra A and their morphisms, together with the law of composition of morphisms as maps, form a *category* that the author denotes by A-algmod. In special cases, the mappings are, of course, often subjected to appropriate restrictions.

As suggested by the terminology, there is also a notion of a *right* A-module for the space X. The definition is an obvious modification of the definition of a left A-module: simply replace the notation $a\cdot x$ by $x\cdot a$, and assume the identity

$$(x\cdot a)\cdot b = x\cdot(ab).$$

Now, if A and B are any pair of algebras, then X is called an *A-B-bimodule* if it is a left A-module and a right B-module and satisfies the identity

$$(a\cdot x)\cdot b = a\cdot(x\cdot b).$$

If $A = B$, the term A-A-bimodule is replaced by A-bimodule. It is obvious that the vector space of an algebra A falls into each of the three module categories, where the left and right operations of A on its vector space are provided by the given algebra multiplication.

The above sketches of a few items associated with the module concept represent only a tiny glimpse of the multitude of ideas and techniques that are being used in the modern study of Banach algebras and their generalizations. They already suggest, however, how such developments tend to shift the emphasis in the old point of view and even transform a familiar subject into an essentially new discipline with a significantly different intuitive content. The situation reminds me of an incident many years ago that involved an eminent *classical* analyst who was able to "master" the field of functional analysis and make important contributions to it. At one point, I was asked to read something he had written that contained material from both fields, and I could not help noticing at the time the difference in their treatments. The classical material flowed so smoothly that it gave an impression of "flying", while the other, though very clear and well written, did not "take off" in the same way. The difference, of course, was not in the nature of the mathematics, but rather in how the author thought about the two subjects. My own experience with some of the material in the present book is quite analogous to the above. I am able to work carefully through a batch of formalities and wind up with an understanding of the underlying mathematics but must confess an inability to "fly" with it, something that would probably be automatic for a present-day young mathematician.

The Helemskii book is devoted ultimately to a study of certain *generalizations* of Banach algebras, though it also contains a great deal of introductory material on Banach algebras proper as well as some elementary algebra. It is worth recalling that the original appearance of Banach algebras coincided with the 1939 publication in Doklady of I. M. Gelfand's report of his work on "normed rings". A normed ring was actually a complete normed *algebra*, which meant that the underlying vector space of the algebra was a Banach space with a norm under which multiplication was continuous. Continuity was usually guaranteed by the "multiplicative condition" that required $\|xy\| \leq \|x\| \, \|y\|$ for all elements x and y in the algebra. The name, Banach algebra, came into use around 1945.

The importance of Banach algebras was that they provided a setting for bringing together in a substantial way the fields of algebra and analysis. There were, of course, many examples from classical and functional analysis that involved multiplication operations, so that the systems under study were actually *algebras* in the technical sense. At the same time, many analysts tended to regard multiplication as a kind of computational device rather than as a key structure property of the system being studied, so the transition to a genuine algebra point of view was not always complete. Things were further complicated at the time by the fact that the fields of algebra and analysis had for some years developed quite independently of one another with little transfer between them. For example, many analysts were not familiar with the simple concept of an "ideal" in algebra. Analogous problems are more or less inevitable in similar situations and, as already suggested, arise in connection with some of the material in the Helemskii book.

The Banach algebra generalizations that are of interest in this book may be obtained from Banach algebras by replacing the usual Banach space norm topology by something more general. One approach is to replace the norm by one or more seminorms. Recall that a seminorm satisfies all of the properties of a norm, except that it may be zero on some nonzero elements of the space. It may therefore be used to define a topology on a vector space E exactly as in the case of a norm. In particular, a basis for the topology consists of sets of the form

$$\{x \in E \colon \|x - x_0\| < \varepsilon\},$$

where ε is an arbitrary positive number and x_0 is an arbitrary vector in E. Incidentally, it is easy to see that these basis sets are linearly convex.

If a vector space E is provided with a *family* of seminorms, then it is said to be "polynormed" and carries a topology with a basis consisting of all of its seminorm basis sets. Two families of seminorms are said to be "equivalent" iff they determine the same topology on E. A topology will be Hausdorff iff the seminorms in the given family can be simultaneously zero only on the zero vector, and the space is "complete" provided Cauchy nets converge. If the family of seminorms is countable, then the defined topology is actually determined by a metric so the space, when complete, is a "Frechet space".

If the polynormed space E is an algebra, then the question of continuity of multiplication arises. This concerns continuity of the mapping,

$$E \times E \to E, \qquad (x, y) \mapsto xy,$$

and can involve either joint or separate continuity. Joint continuity means that the mapping from $E \times E$, as a topological product space, to the space E is continuous. Separate continuity means that the two maps, $x \mapsto xy$ (y fixed) and $y \mapsto xy$ (x fixed), of E into itself are continuous with respect to the topology in E.

A convenient way to ensure joint continuity is to require the seminorms to be multiplicative, as defined above, in which case E is called a "multiplicatively polynormed" algebra. If the topology is also Hausdorff and complete, the author calls the algebra an "Arens-Michael algebra" in honor of R. Arens and E. A. Michael, who studied them.

Although many authors would automatically assume that a polynormed algebra be multiplicatively polynormed and therefore have jointly continuous multiplication, Helemskii, except in special cases, only requires that multiplication in a polynormed algebra be separately continuous. Because basic neighborhoods are convex, polynormed algebras are also called "locally convex" algebras, which explains the term used in the title of the book. As the author points out, the two concepts are actually equivalent.

Certain polynormed algebras are in many respects the most immediate generalization of Banach algebras, and a number of the results for Banach algebras have been extended in one form or another to them. A systematic presentation of such results is a basic motivation for this book, though it is by no means limited to them.

In closing, I must admit some lack of enthusiasm for the general organization of material in the book. For example, I find it a bit difficult to imagine the nature of a student's background that would enable him to benefit fully from the foundations in Chapter 0, and the order in which some topics in the main text are presented sometimes appears artificial. On the other hand, some of these criticisms tend to be eased by the fact that the book is well equipped with an extensive bibliography and a substantial index of notation as well as a good general index. Furthermore, the author is very careful about including in the text needed references to both the literature and to sources of concepts and notation within the text itself. A diligent student who is willing to work systematically through the book might very well wind up an expert in the subject, though perhaps with a very different perspective than someone previously exposed to such material.

C. E. RICKART
YALE UNIVERSITY
E-mail address: RICKART@LOM1.MATH.YALE.EDU.

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Catalan's conjecture, by Paulo Ribenboim. Academic Press, New York, 1994, xv + 364 pp., $64.95. ISBN 0-12-587170-8

Catalan conjectured in 1844 that the only solution to the equation

$$x^p - y^q = 1 \qquad (*)$$

in integers x, y, p, q, all > 1, is given by $3^2 - 2^3 = 1$; in other words, 8 and 9 are the only consecutive powers. As Dickson pointed out in his famous *History of the theory of numbers*, in the Middle Ages Levi ben Gerson had solved the case $x = 3$, $y = 2$, and in 1738 Euler had solved the case $p = 2$, $q = 3$. Catalan himself contributed little, essentially only a simple remark on $x^y - y^x = 1$. Nevertheless, the conjecture gained considerable notoriety, and it became plain that it presented a challenge to number theorists somewhat akin to Fermat's Last Theorem.

As with the Fermat problem, factorization techniques over the cyclotomic and other fields have shed light in particular instances. Thus in 1850 V. A. Lebesgue dealt with the case $x^p - y^2 = 1$, and in 1964 Chao Ko treated the more difficult example $x^2 - y^q = 1$; this included an earlier theorem of S. Selberg with $p = 4$. The equations $x^3 - y^q = 1$ and $x^p - y^3 = 1$ were successfully resolved by Nagell in 1921, and the work led to valuable advances, notably by Ljunggren, on related equations such as $(x^p - 1)/(x - 1) = y^q$. Moreover, in 1961 Cassels obtained a particularly striking result in this context; namely, if p, q are odd primes, as one can assume, then $(*)$ implies that p divides y and q divides x. As Makowski noted, this shows that we cannot have three consecutive integer powers.

One of the most remarkable applications of the theory of linear forms in logarithms has been the effective determination of an explicit bound for all solutions x, y, p, q of $(*)$. The result depends on estimates for a nonvanishing expression

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n,$$

where the b's are integers and the α's are algebraic numbers. The original work of the reviewer of 1966 furnished a lower bound for $\log|\Lambda|$ that varied as a power of $\log B$, where B is the maximum of the $|b|$'s. This led to a new and effective resolution of the Thue equation whence, on viewing $(*)$ as superelliptic and reducing to Thue type, it followed that all solutions x, y are bounded by an explicit expression involving only p and q. The really decisive step came in the early 1970s with the Sharpening series of papers; here it was shown that the dependence of the lower bound for $\log|\Lambda|$ on A, the maximum of the