their regularity and, ideally, to show that the weak solutions in fact are smooth and satisfy (0.2) classically. As a first method in this regard, the L^2-theory, or Hilbert space method, is presented, yielding smoothness of solutions of linear equations with smooth coefficients. More refined estimates are available from Schauder theory. In Giaquinta's book, this theory is developed based on ideas of Morrey and on Campanato's perturbation method—that is, free of potential theory—yielding Hölder continuity of a solution (and its first derivatives) to a linear elliptic partial differential equation

$$-\frac{\partial}{\partial x^\alpha}\left(a^{\alpha\beta}(x)\frac{\partial}{\partial x^\beta}u\right) = \frac{\partial}{\partial x^\alpha}f^\alpha$$

with Hölder continuous coefficients $a = (a^{\alpha\beta})$ and right-hand side $f = (f^\alpha)$. In the fourth chapter the L^p-estimates are presented, again avoiding potential theory but being based on an interpolation theorem of Stampacchia and the results of the two preceding chapters instead. Chapter 5 contains some beautiful new material related to the regularity theorem of De Giorgio-Nash and Moser's Harnack inequality; in particular, with the aid of a covering lemma due to Krylow-Safonov and some results of Di Benedetto-Trudinger, it is shown that Harnack's inequality holds for functions in the De Giorgi class. Finally, these ideas are applied to obtain partial regularity of minimizers of functionals of type (0.1), in particular, partial regularity of energy-minimizing harmonic maps.

The book summarizes the contents of a series of lectures given by Giaquinta in the Nachdiplom graduate program of the ETH Zürich in 1983-84. It is similar to notes by Giaquinta, published by Princeton University in 1983; however, it reaches beyond the latter in many important respects, for instance, regarding the regularity results for functions in De Giorgi's class mentioned above. The Birkhäuser series "Lectures in Mathematics ETH Zürich" was recently created as a means to make the notes of the ETH "Nachdiplom" lectures available to the mathematical community.

Giaquinta is an excellent expositor. Moreover, he has mainly contributed to the developments outlined in his notes. Anyone interested in partial differential equations would want to have this book on the shelf.

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1. *Composition operators and classical function theory*, by Joel H. Shapiro. Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993, xvi + 223 pp., $34.00. ISBN 0-387-94067-7 2. *Composition operators on function spaces*, by R. K. Singh and J. S. Manhas. North-Holland Mathematics Studies, vol. 179, North-Holland, Amsterdam, 1993, x+315 pp., 200 Dfl. ISBN 0-444-81593-7.

If F is a space of functions defined on a domain D and ϕ is a function taking D into itself, then the composition operator induced by ϕ, denoted $\mathscr{C}_\phi$, is defined by $(\mathscr{C}_\phi f)(z) = f(\phi(z))$ for $f \in F$ and $z \in D$.

Composition of functions is a fundamental operation in all of mathematics, of course, and composition operators on spaces of measurable functions are the basic objects in classical ergodic theory. However, the explicit study of composition operators as linear operators on Banach spaces is of relatively recent origin.

The subject has two main branches: the analytic, typified by the study of composition operators on the Hardy space $\mathscr{H}^2$ of analytic functions on the unit disk with square-summable Taylor coefficients, and the measure-theoretic, concerning composition operators on $\mathscr{L}^p$ spaces.

Eric Nordgren's 1968 paper [7] can be said to have initiated the subject; it contained results in both branches. Nordgren's work stimulated the doctoral dissertation of Howard Schwartz [12] on the analytic case and of Nordgren's student R. K. Singh [14] on the measure-theoretic case. In 1977 Nordgren gave several lectures on the subject at a conference in Long Beach, California. These lectures and the expanded version, which was published as an expository paper [8], created broad interest in composition operators; there are numerous papers listed in the bibliographies of the books under review.

Shapiro's book treats the analytic case. If ϕ is an analytic function mapping the open unit disk in the complex plane into itself, then $\mathscr{C}_\phi$ is studied as an operator on $\mathscr{H}^2$. Shapiro assumes only the most basic facts about complex analysis and operators on Hilbert space and then simultaneously develops the additional function theory and operator theory he requires. This injects new life into many classical results in complex analysis.

Shapiro's book emphasizes his own research; fortunately, this includes some very interesting material. The seemingly innocuous question of which $\mathscr{C}_\phi$ are compact has generated a number of fascinating theorems. Schwartz [12] obtained some preliminary results; the general question proved to be quite difficult. In the case where ϕ is univalent, $\mathscr{C}_\phi$ is compact if and only if ϕ does not have an angular derivative at any point of the unit circle. This was established by MacCluer and Shapiro [5] and is nicely described in Shapiro's book. The proof requires the Julia-Caratheodory Theorem on angular derivatives, which Shapiro beautifully explains based on ideas of Valiron [15]. (As Shapiro indicates, Sarason [11] has found a Hilbert-space proof of the Julia-Caratheodory Theorem.)

The problem of characterizing compactness when ϕ is not necessarily univalent involves another important concept from complex analysis, the Nevanlinna counting function. This function, denoted $\mathscr{N}_\phi(w)$, was introduced [6] to study the distribution of values of analytic functions. Shapiro [13] proved that $\mathscr{C}_\phi$ on $\mathscr{H}^2$ is compact if and only if

$$\lim_{|w|\to 1^-} \frac{\mathscr{N}_\phi(w)}{\log|\frac{1}{w}|} = 0.$$

A leisurely and carefully explained discussion of the proof is contained in Shapiro's book.

The Denjoy-Wolff Theorem and related results on iteration of analytic functions play an important role in the study of composition operators on $\mathscr{H}^2$. This theorem is also well presented, although most of its applications to composition operators are only briefly referred to in end-of-chapter notes. A very nice study of cyclicity and hypercyclicity of composition operators, due to Bourdon and Shapiro [1], is described in detail.

Shapiro's book is an attractive introduction to the subject of composition operators on spaces of analytic functions. The remarkable interplay between function-theoretic and operator-theoretic concepts is beautifully illustrated throughout the book. The writing is clear and engaging. Each chapter contains a number of interesting exercises. It would be an excellent text for a graduate student to use for a reading course.

The book by Singh and Manhas is quite different from that of Shapiro. While Shapiro gives a self-contained introduction to a part of the analytic theory, Singh and Manhas give a broad survey of the larger subject. The treatment of composition operators in $\mathscr{L}^p$ spaces is quite complete. There is also an extensive study of composition operators on various spaces of continuous functions.

Singh and Manhas only outline the analytic cases, but they give references to the study of composition operators on various spaces of analytic functions of one and several variables. Their last chapter, entitled "Some Applications of Composition Operators", contains discussions of several topics, including the well-known theorems characterizing isometries of certain function spaces as products of composition and multiplication operators. There is a description of some of the basic results of ergodic theory and of certain dynamical systems.

The book by Singh and Manhas contains a comprehensive bibliography which will be useful to anyone who wishes to study composition operators.

There is much additional material about composition operators which is ready for exposition but has not been included in either of the above books. Two of the other leading workers in the field, Carl Cowen and Barbara MacCluer, are jointly writing a book about composition operators on spaces of analytic functions. This book will likely include the treatment of composition operators on the Bergman and Dirichlet spaces and a thorough discussion of what is known about the spectra of composition operators on spaces of analytic functions of one and several variables, an interesting topic to which Cowen and MacCluer have made major contributions.

Pending the publication of Cowen-MacCluer, there are two good expository articles that supplement the books under review: Cowen's article [2] gives a general survey and many references; Wogen's exposition [16] gives an overview of composition operators on spaces of analytic functions of several variables.

Although many results have been obtained, there are still a number of very interesting questions about composition operators. There is much more to be learned about spectra in various settings. Commutants of composition operators seem to be very difficult to characterize. Only a little is known about their reducing subspaces (cf. [3]) and thus of the von Neumann algebra generated by a composition operator. There has been no work on $\mathscr{C}^*$ algebras generated by composition operators. Determining the invariant subspaces of a composition operator on $\mathscr{H}^2$ induced by a hyperbolic linear fractional transformation of the disk onto itself is too difficult (the operator has translates that are universal, and thus its invariant subspace lattice contains sublattices isomorphic to each invariant subspace lattice of an operator on separable Hilbert space [9]). However, it is conceivable that the invariant subspace problem could be solved by studying the invariant subspaces of such operators ([9, 10]; a small beginning is made in [4]).

The subject is flourishing. The books under review will make it much easier for newcomers (and oldtimers) to participate in future developments.

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Harmonic maps and minimal immersions with symmetries, by James Eells and Andrea Ratto. Annals of Mathematics Studies, vol. 130, Princeton University Press, Princeton, NJ, 1993, 228 pp., $19.95. ISBN 0-691-10249-X.

A harmonic map between Riemannian manifolds generalizes the notion of a harmonic function: if the target manifold is viewed as embedded in some Euclidean space, then a map $\varphi: M \to N \subset \mathbb{R}^k$ is harmonic if the Laplacian of φ (as a map into $\mathbb{R}^k$) is perpendicular to N. This is precisely the Euler-