BOOK REVIEWS

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Commensurabilities among lattices in PU(1, n), by Pierre Deligne and G. Daniel Mostow. Annals of Mathematics Studies, no. 132, Princeton University Press, Princeton, NJ, 1993, 183 pp., $19.95. ISBN 0-691-00096-4

In the book under review, Deligne and Mostow study a number of different constructions of lattices in $\mathrm{PU}(1, n)$, $n \geq 1$, a lattice Γ being by definition a discrete subgroup of $\mathrm{PU}(1, n)$ with quotient $\mathrm{PU}(1, n)/\Gamma$ of finite Haar measure. Of particular interest are the non-arithmetic lattices in $\mathrm{PU}(1, n)$, $n > 1$. As pointed out on page 3 of §1, the approaches used to date to construct possibly non-arithmetic lattices involve examples of Mostow [M1] of certain subgroups of $\mathrm{U}(1, n)$ generated by complex reflections, monodromy groups of Appell-Lauricella hypergeometric functions, and covering groups of orbifold ball-quotients coming from line arrangements in the projective plane. The lattices constructed by Mostow in [M1] were shown by him [M2] to be close to the Appell-Lauricella lattices, which in turn are related to the complete quadrilateral arrangement in the projective plane: the outcome being that many of the lattices constructed so far are commensurable to the Appell-Lauricella ones. Moreover, as shown in [BHH], with the construction of a larger class of examples in the book under review, a number of lattices arising from line arrangements other than the complete quadrilateral are commensurable to Appell-Lauricella monodromy groups. In all, then, many of the constructions to date are linked to the theory of hypergeometric functions of several variables.

To recall some of the history of hypergeometric functions, Euler [E] introduced the series

$$(1) \qquad F = F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \qquad |x| < 1,$$

where $(a, n) = \frac{\Gamma(a+n)}{\Gamma(a)}$ and a, b, c are any complex numbers with c neither 0 nor a negative integer. He had introduced this series as a solution to the differential equation

$$(2) \qquad x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby = 0$$

and knew that when the real parts of a and $c - a$ are positive, one has the integral representation

$$(3)\quad \begin{aligned} F(a, b; c; x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b} du \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty u^{b-c}(u-1)^{c-a-1}(u-x)^{-b} du. \end{aligned}$$

By studying transformations which leave the integral representation unchanged, Euler derived three transformation formulae, for example:

$$(4)\qquad F(a, b; c; x) = (1-x)^{-a} F\left(a, c-b; c; \frac{x}{x-1}\right).$$

Gauss [Ga] dubbed the $F(a, b; c; x)$ "hypergeometric series". This series has an analytic continuation outside its circle of convergence $|x| = 1$ to the complex plane minus the segment $[1, \infty)$ of the real axis; and this extended function, also denoted by $F(a, b; c; x)$, is called the hypergeometric function of Gauss. For $\operatorname{Re}(a)$, $\operatorname{Re}(c-a) > 0$ the integral representation (3) still remains valid [J] (and gives the analytic continuation of the series in (1)); and if neither a nor $c-a$ is a positive integer, one may replace the line integral by an integral over a Pochhammer cycle around 0 and 1 [Po1, Po2]. In [K] Kummer gave the complete table of the 24 solutions of (2) obtained by applying the three transformation formulae of Euler to six series solutions. These six series solutions are given by a certain choice of two solutions, valid for non-integral exponent differences (that is, for non-integral c, $c-a-b$, $a-b$) in a neighbourhood of each of $0, 1, \infty$ and generating the solution space there. For example, if c is not an integer, the solution space of (1) is generated in $|x| < 1$ by F and the series

$$x^{1-c} F(a+1-c, b+1-c; 2-c; x).$$

One can similarly recover Kummer's complete table in terms of the integral representation of these six series given (up to constants) for $g, h \in \{0, 1, \infty, x\}$ by the integrals

$$(5)\qquad \int_g^h u^{b-c}(u-1)^{c-a-1}(u-x)^{-b} du$$

(Hermite [H], Pochhammer [Po1], Schläfli [S]), of which (3) is a special case. Just as for F the 24 series of Kummer have a continuation for which their integral representations remain valid. In their common domain of continuation, on choosing a branch, between any three of the six integrals of (5) there exists a linear relation with constant coefficients.

In 1857 Riemann [R] characterised the differential equation (2), which has singularities only at the regular singular points $0, 1, \infty$, in terms of the branching data of its solutions at these points under the assumption that the exponent differences were not integral (so eliminating logarithmic terms). This problem of Riemann was also treated by Fuchs [F]. The cases where some of the exponent differences are integral have been studied (see for example [AS, p. 563]). For example, for $c = 1$, two independent solutions of (2) are given for $|x| < 1$ by the series

$$\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(n!)^2} x^n,$$

$$\left(\sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(n!)^2} x^n\right) \log(x) + \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(n!)^2} c(a,b,n) x^n$$

for constants $c(a, b, n)$ (given, for example, in [AS, p. 564]).

In order to determine for which values of a, b, c the hypergeometric function is an algebraic function of x (Schwarz's list), Schwarz [Sch] was led to consider the monodromy group of (2): the required values of a, b, c are those for which (2) has finite monodromy group. As the monodromy group leaves invariant the solution space of (2), this leads naturally to considering its representation in $\mathrm{PGL}(2, \mathbb{C})$. Schwarz also determined criteria for the developing map, given by the ratio of two linearly independent solutions of (2), to be invertible to a single-valued function. In this case the monodromy group has a fundamental domain and is discrete in $\mathrm{PGL}(2, \mathbb{C})$: it is a triangle group generated by even numbers of reflections in the sides of a spherical, euclidean, or hyperbolic triangle. (In terms of the parameters a, b, c the angles of this triangle are $\pi \mid 1-c \mid, \pi \mid c-a-b \mid, \pi \mid a-b \mid$.) In the first case the monodromy group is a finite subgroup of $\mathrm{PGL}(2, \mathbb{C})$, in the second case a subgroup of $\mathrm{GL}(1, \mathbb{C}) \ltimes \mathbb{C}$, and in the third case a subgroup of $\mathrm{PU}(1, 1)$ acting on the 1-ball or disc. Schwarz proved that a triangle group whose defining triangle has angles of the form $\pi/p, \pi/q, \pi/r$ for positive (possibly infinite) integers p, q, r always has a discontinuous action. However, the converse may not hold: examples already occur in Schwarz's list [CoWo2] in the spherical case. In 1926 Appell and Kampé de Fériet published a book [AKdF] containing a lengthy study of hypergeometric functions of several variables. To quote the introduction of their book: "*Dans la première partie nous exposons l'ensemble des résultats relatifs aux fonctions hypergéométriques de plusieurs —et plus spécialement de deux— variables.*" This book, which brings under one roof most of what was then known about hypergeometric functions in several variables, was largely inspired by an 1882 memoir of Appell [A] in which he defines four series in two variables generalising the Gauss hypergeometric function. Each one of Appell's double series satisfies a system of two linear second-order partial differential equations. The solution spaces of these systems of partial differential equations have dimension (or rank) 4 for three of the systems and 3 for the remaining one (see [Y2, p. 62]). The extension of the results of Appell to the n-variable case was done by Lauricella [L], whose results also appear in [AKdF]. Work of Goursat [Go] and Picard [P1a, P1b] in the early 1880s complemented the work of Appell and showed that the two-variable analogue of the classical Riemann problem leads naturally to one of the above functions, namely, the Appell function

$$\begin{aligned} F_1 &= F_1(a, b, b'; c; x, y) \\ &= \sum_{m,n=0}^{\infty} \frac{(a, m+n)(b, m)(b', n)}{(c, m+n)} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad |x| < 1, |y| < 1, \end{aligned}$$

arising in the rank 3 case. In his 1893 thesis Le Vavasseur [LeV] made a very explicit study of the integrals of the system of partial differential equations for F_1 and the relations between them. Moreover, Picard [P1a, P1b] characterised the solutions of the F_1 system as the multivalued functions of two variables with exactly three linearly independent branches and with prescribed ramification along the seven lines: $x, y = 0, 1, \infty$ and $x = y$, of which F_1 is the only solution holomorphic and taking the value 1 at the point $(x, y) = (0, 0)$. Generalising Schwarz's work, Picard [P1a, P1b, P2a, P2b] also found criteria for the monodromy group of the F_1 system to be a discrete subgroup of $\mathrm{PGL}(3, \mathbb{C})$. However, from the modern mathematical point of view, there are inadequacies in Picard's treatment, and one of the contributions of Deligne and Mostow's monumental 1986 papers [DM, M2] is to correct Picard's proof using methods from algebraic geometry and to prove an analogous criterion for arbitrary dimension. Terada had formulated and proved similar results using function-theoretic techniques in important papers [Te1, Te2], but he did not obtain all the Deligne-Mostow results and his techniques were not as powerful. In a valuable and interesting book Yoshida [Y2] develops the theory of Fuchsian differential equations, most especially that of the hypergeometric ones, emphasising the link with orbifolds which is central to the book under review. Using the work of Hirzebruch and Höfer [BHH], Yoshida [Y2] employs differential geometry to prove a version of the Deligne-Mostow [DM] results using the connection between hypergeometric functions and the orbifold given by prescribing ramifications on the complete quadrilateral line arrangement in $\mathbb{P}_2$.

The lattice criteria of Deligne and Mostow's 1986 papers are as follows. They show [DM, Theorem 11.4, p. 66]: For an integer $n \geq 1$, let $\mu = (\mu_1, \dots, \mu_{n+3})$ be an $(n+3)$-tuple of real numbers with $0 < \mu_s < 1$ for $s = 1, \dots, n+3$ and such that $\sum_{s=1}^{n+3} \mu_s = 2$; call this a ball $(n+3)$-tuple. Then, if INT: for all $s \neq t$ such that $\mu_s + \mu_t < 1$ we have $(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z}$; then Γ_μ is a lattice in $\mathrm{PU}(1, n)$. Here Γ_μ is the monodromy group in $\mathrm{PGL}(n+1, \mathbb{C})$ of the Appell-Lauricella system of linear partial differential equations of rank $n+1$ whose solution space $V(\mu)$ is generated by integrals of the form

$$\int_g^h \left\{ \prod_{i=1}^n (u - x_i)^{-\mu_i} \right\} u^{-\mu_{n+1}} (u-1)^{-\mu_{n+2}} du$$

where $g, h \in \{0, 1, \infty, x_1, \dots, x_n\}$. The Appell function F_1 is a constant multiple of an integral of this form with $n = 2$ and is given for $\mathrm{Re}(a)$, $\mathrm{Re}(c-a) > 0$ by

$$\begin{aligned} &F_1(a, b, b'; c; x, y) \\ &\quad = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty u^{b+b'-c}(u-1)^{c-a-1}(u-x)^{-b}(u-y)^{-b'} du. \end{aligned}$$

Notice that on setting $y = 0$, we recover the formula (3) for F. As pointed out in [DM, §14.2 Case A, p. 82] it is easy to verify that, for $n > 2$, if μ satisfies INT, then $\mu_s + \mu_t < 1$ for all $s \neq t$. One can check directly from the list of μ satisfying INT when $n = 2$ [DM, p. 86] that for all of them $(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z}$, $s \neq t$, also when $\mu_s + \mu_t > 1$. This fact follows easily also for $n = 1$. Hence as it turns out, for $n \geq 1$ the condition INT is equivalent to: for all $s \neq t$, $(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z} \cup \{\infty\}$. In [M2, §2, Theorem], Mostow deduced the same result with INT replaced by the weaker ΣINT: there is a subset S_1 of $S = \{1, \dots, n+3\}$ such that for all $s, t \in S_1$ we have $\mu_s = \mu_t$; and for all $s, t \in S$, $s \neq t$, such that $\mu_s + \mu_t < 1$, we have $(1 - \mu_s - \mu_t)^{-1} \in (\frac{1}{2}\mathbb{Z})$ if $s, t \in S_1$ and $(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z}$ otherwise. Mostow showed therefore that if μ satisfies ΣINT, then Γ_μ is a lattice in $\mathrm{PU}(1, n)$. It is easy to check that ΣINT is equivalent to the condition: for all $s, t \in S$, $s \neq t$ such that $\mu_s + \mu_t < 1$ we have $(1-\mu_s-\mu_t)^{-1} \in (\frac{1}{2}\mathbb{Z})$ if $\mu_s = \mu_t$, and $(1-\mu_s-\mu_t)^{-1} \in \mathbb{Z}$ if $\mu_s \neq \mu_t$. In [M3, pp. 584–586] Mostow gives the list (calculated on computer by Thurston) of all ball $(n+3)$-tuples, $n \geq 2$, satisfying ΣINT. Using this list, one can check directly that for $n \geq 2$, if μ satisfies ΣINT, then $(1-\mu_s-\mu_t)^{-1} \in \mathbb{Z}$ for $s \neq t$ with $\mu_s + \mu_t > 1$. This obviously fails for $n = 1$, and there are even ball 4-tuples satisfying ΣINT but with $0 > (1-\mu_s-\mu_t)^{-1} \notin \mathbb{Z}$ and $\mu_s \neq \mu_t$ (see [M3, Theorem 3.8, p. 570, example $D'_{[p,q]}$, p integer, q odd integer]). For $n \geq 2$ the condition ΣINT is therefore equivalent to (compare with [CoWo3, p. 668]): for all $s, t \in S$, $s \neq t$, we have $(1-\mu_s-\mu_t)^{-1} \in (\frac{1}{2}\mathbb{Z}) \cup \{\infty\}$ if $\mu_s = \mu_t$, and $(1-\mu_s-\mu_t)^{-1} \in \mathbb{Z} \cup \{\infty\}$ if $\mu_s \neq \mu_t$.

In the case $n = 1$, Schwarz's discreteness condition for a triangle group defined by a hyperbolic triangle with angles of the form $\pi/p, \pi/q, \pi/r$ is equivalent to the condition INT for the ball 4-tuples $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ and $\mu' = (1-\mu_1, 1-\mu_2, 1-\mu_3, 1-\mu_4)$ where

$$\mu_1 = \frac{1}{2}\left(1 - \frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right), \qquad \mu_2 = \frac{1}{2}\left(1 + \frac{1}{p} - \frac{1}{q} + \frac{1}{r}\right),$$
$$\mu_3 = \frac{1}{2}\left(1 + \frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right), \qquad \mu_4 = \frac{1}{2}\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right).$$

In fact Γ_μ and $\Gamma_{\mu'}$ are conjugate in PU(1, 1) to the above triangle group. See [DM, 14.3] and [M3, p. 570]. In [M3, Theorem 3.8, p. 570] Mostow gives the list of all the ball 4-tuples μ with Γ_μ discrete in PU(1, 1) where μ does not satisfy INT. He obtains this list from his list [M3, Theorem 3.7, p. 569] of hyperbolic triangles whose angles are not of the form $\pi/p, \pi/q, \pi/r$ for $p, q, r \in \mathbb{Z} \cup \{\infty\}$ but whose triangle groups are discrete in PU(1, 1). This list is also to be found in [Kn, p. 297], although Knapp's list apparently has the additional member $2\pi/7, \pi/7, \pi/3$ that is $\mu = (25/42, 31/42, 23/42, 5/42)$.

For the case $n = 2$, the apparently stronger condition for a ball 5-tuple that for all $s \neq t$ one has $(1-\mu_s-\mu_t)^{-1} \in \mathbb{Z} \cup \{\infty\}$ is Picard's original condition that Γ_μ be a lattice in PU(1, 2). In [LeV] the list of Le Vavasseur contains all the 27 ball 5-tuples (up to permutation) satisfying Picard's condition. (Picard actually formulated his discreteness condition without the assumption that μ be a ball 5-tuple, and Le Vavasseur gave a complete list of this larger class of μ; see [DM, p. 87–88].) From our previous remarks Picard's condition is equivalent to the INT condition (see also [DM, §15, p. 87]). For $n = 2$, we know from Mostow's list [M3, pp. 584–586] that there are 53 ball 5-tuples (up to permutation) satisfying ΣINT, of which 14 correspond to non-arithmetic Γ_μ.

In [M3] Mostow determined when the converse of the criterion of [M2] holds and studied lattices of the family Γ_μ violating ΣINT, which occur only for $n \leq 3$. For $n = 3$ there is only one exceptional Γ_μ up to commensurability, corresponding to

$$\mu = (1/12, 3/12, 5/12, 5/12, 5/12, 5/12).$$

For $n = 2$ there are nine 5-tuples μ not satisfying ΣINT with projective monodromy group Γ_μ which is discrete (and hence a lattice [M3, Proposition 5.3, p. 580]) in PU(1, 2). Five of these nine exceptions are arithmetic (as is the exceptional μ for $n = 3$): for example, the case

$$\mu = (1/12, 3/12, 5/12, 5/12, 10/12).$$

In [Sa, §3, Theorem 3.1; §4, Theorem 4.1] Sauter proved a conjecture of Mostow [Sa, p. 348] to the effect that for $n = 2$ the nine exceptional Γ_μ not satisfying $\Sigma\,\mathrm{INT}$ are commensurable to a Γ_ν where ν satisfies $\Sigma\,\mathrm{INT}$.

In the present book Deligne and Mostow pursue their study of lattices in $\mathrm{PU}(1, n)$ and present the latest developments in a relatively accessible style, dedicating a number of chapters to orbifolds arising from arrangements of curves on algebraic surfaces in the spirit of [BHH]. They extract the best aspects of the previous techniques of algebraic and differential geometry mentioned above together with function theory, giving an overall coherent presentation yielding new results, with an emphasis on identifying when two lattices in $\mathrm{PU}(1, n)$ are commensurable. The commensurability results in the book all consist of showing that the lattice of interest in $\mathrm{PU}(1, n)$ is commensurable to a certain $\Gamma_{\mu,H}$ for μ a ball $(n+3)$-tuple satisfying $\Sigma\,\mathrm{INT}$, where H is a subgroup of the symmetric group $\Sigma(n+3)$ on $n+3$ letters leaving μ invariant. Indeed, let $Q = \{(x_1, \dots, x_n) \in \mathbb{P}_1^n \mid x_i \neq x_j, i \neq j, x_i \neq 0, 1, \infty\}$. Then the solutions of the Appell-Lauricella system of partial differential equations define multivalued functions on Q. The group $\Gamma_{\mu,H}$ is the projective monodromy group of a local system of functions, induced by an H-invariant twist of $V(\mu)$, on Q'/H where (for $n \geq 2$) Q' is the open subset of Q where H acts freely. Such a twist is obtained by multiplying the elements of $V(\mu)$ by the same multivalued function and corresponds to assuring the symmetry with respect to H: compare with the effect of the transformation described in (4). Indeed, as Euler had found in dimension 1, when one wishes to take account of permutations of the parameters μ_i, one is forced to consider twists of hypergeometric functions, which Deligne and Mostow call hypergeometric-like local systems. Generally speaking, a local system on a connected analytic variety X is made up of the constant coefficient linear combinations of the branches of a multivalued holomorphic function on X whose branches at any point of X span a finite-dimensional vector space. Sections 2 through 7 of the book develop a theory of such hypergeometric-like local systems on Q, characterising them by studying their local properties on a partial compactification Q^+ of Q. In view of the fact that one often wishes to restore the symmetry between the $s \in S = \{1, \dots, n+3\}$, it is convenient to describe Q as the quotient $M/\mathrm{PGL}(2, \mathbb{C})$ where $\mathrm{PGL}(2, \mathbb{C})$ acts diagonally on

$$M = \{(x_1, \dots, x_{n+3}) \in \mathbb{P}_1^{n+3} \mid x_i \neq x_j, i \neq j\}.$$

Then M^+ is defined as the space of $(n+3)$-tuples $(x_1, \dots, x_{n+3})$ with $x_s = x_t$ for at most two elements s, t of S, and Q^+ is defined as the quotient $Q^+ = M^+/\mathrm{PGL}(2, \mathbb{C})$. In §2 of the book various results about divisors on algebraic varieties over $\mathbb{C}$ are explained for use in the remainder of the book. In §3 it is shown that multivalued functions $f = \prod_i f_i^{\alpha_i}$ with $\alpha_i \in \mathbb{C}$ and f_i invertible regular functions on Q are uniquely determined by their branching data along $D_{s,t}$, the image in Q^+ of $x_s = x_t$, $s \neq t$, $s, t \in S$. The determinations of the multivalued function f are constant multiples of each other and hence span a local system of rank 1. In §4 local systems of holomorphic functions and in particular Appell-Lauricella hypergeometric functions on Q are treated, and in §5 the relation between hypergeometric-like local systems and Gelfand's hypergeometric local systems on a Zariski open subset of the $(n+3) \times 2$ matrices is described [G]. This leads to an efficient derivation of the Appell-Lauricella system of partial differential equations. A local system V on Q of dimension $n+1$ has exponents $(\alpha_{s,t}, \beta_{s,t})$, $s \neq t$, along $D_{s,t}$ if in a neighbourhood of $D_{s,t}$ the local system V is a direct sum $V = V' \oplus V''$ with $\dim V' = 1$ and $\dim V'' = n$ and if, for z a local equation for $D_{s,t}$ in a neighbourhood $U(p)$ of $p \in D_{s,t}$, the local systems $z^{-\beta_{s,t}} V'$ and $z^{-\alpha_{s,t}} V''$ extend across $D_{s,t}$ as local systems of holomorphic functions $(z^{-\beta_{s,t}} V')_{U(p)}$ and $(z^{-\alpha_{s,t}} V'')_{U(p)}$ on $U(p)$ (see §6.6). Hence there are holomorphic functions g_i, $i = 0, \dots, n$, defined in $U(p)$ such that locally on $U(p) - D_{s,t}$ the functions $e_0 = z^{\beta_{s,t}} g_0$ and $e_i = z^{\alpha_{s,t}} g_i$, $i = 1, \dots, n$, form a basis of V. One always supposes that $\alpha_{s,t} - \beta_{s,t} \notin \mathbb{Z}$. The notion of strict (non-degenerate) exponents of an $(n+1)$-dimensional, $n = \dim Q$, local system of holomorphic functions on Q is defined in §6. For $(\alpha_{s,t}, \beta_{s,t})$ to be strict exponents means that neither g_0 nor any non-trivial constant coefficient linear combination of the g_i, $i = 1, \dots, n$, vanishes everywhere on $D_{s,t}$. Finally, the result of §7, Theorem 7.1, p. 55 says that any (étale) local system possessing strict exponents $(\alpha_{s,t}, \beta_{s,t})$ along $D_{s,t}$ for all $s, t \in S$, $s \neq t$, is necessarily hypergeometric-like: a twist by a rank 1 local system of $V(\mu)$ for $\mu = (\mu_s)_{s=1}^{n+3}$ where $\beta_{s,t} - \alpha_{s,t} = (1 - \mu_s - \mu_t)$, $s, t \in S$, $s \neq t$. One can think of these exponent differences as arising via the passage from projective to affine coordinates in the developing map, a higher-dimensional analogue of the expression of the Schwarz triangle map as the quotient of two solutions of (2). In particular, the results in the book generalise the work of Riemann ($n = 1$) referred to earlier. This treatment is related to but is also different from Terada's proof [Te1] using function-theoretic methods of a uniqueness theorem for the Appell-Lauricella functions, up to a multiplicative constant, given the exponents. Terada also considers the case of integral exponent differences where logarithmic terms are introduced, as explained above for $n = 1$. Terada's work is an important predecessor of Deligne-Mostow, and the exact relation between the two approaches is described in §7.13. For some of this brief overview of §§2–7 we have borrowed from §1, which provides a succinct introduction to and summary of the book.

The book develops geometric proofs of the type of commensurability result obtained by Sauter, which most importantly yield the commensurability result of Mostow's conjecture for the non-arithmetic non-Σ INT cases which are [M3, 5.5, p. 582]

$$(4/18, 5/18, 5/18, 11/18, 11/18), \quad (4/21, 8/21, 10/21, 10/21, 10/21),$$

$$(5/24, 10/24, 11/24, 11/24, 11/24), \quad (7/30, 13/30, 13/30, 13/30, 14/30).$$

In these geometric proofs one works with the stable compactification of Q: for μ a ball 5-tuple let $\overline{M}_\mu = \{(x_1, \dots, x_5) \in \mathbb{P}_1^5 \mid \sum_{x_i = x_j} \mu_i < 1, j = 1, \dots, 5\}$. Then the stable (partial, in general) compactification of Q is $\overline{Q}_\mu = \overline{M}_\mu / \mathrm{PGL}(2, \mathbb{C})$, and it depends only on the set $\mathcal{T}$ of 2-element subsets $\{s, t\}$ of S with $\mu_s + \mu_t > 1$. If μ is invariant by a subgroup H of $\Sigma(5)$, and if for $\mu_s + \mu_t < 1$ we have: $(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z}$ if the transposition $(st) \notin H$, and $2(1 - \mu_s - \mu_t)^{-1} \in \mathbb{Z}$ if $(st) \in H$, then with $\Gamma_{\mu, H}$ as above

$$B_2 / \Gamma_{\mu, H} \simeq \overline{Q}_\mu / H$$

as orbifolds, with Γ_μ of finite index in $\Gamma_{\mu, H}$. When μ satisfies INT, we can take H to consist of the identity. For $\{s, t\} \notin \mathcal{T}$, along the image of

$x_s = x_t$ in $\overline{Q}_\mu/H$ one has the ramification $(1-\mu_s-\mu_t)^{-1}$ if $(st) \notin H$ and $2(1-\mu_s-\mu_t)^{-1}$ if $(st) \in H$. Let $\overline{Q}_1$ be the compactification corresponding to $\mathscr{T} = \phi$, let $\overline{Q}_2$ be the compactification corresponding to $\mathscr{T} = \{\{1, 2\}\}$, and let $\overline{Q}_3$ be the same for $\mathscr{T} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Let H_1 be generated by the transpositions $(12), (34)$, and let H_2 be generated by (34). Deligne and Mostow show the isomorphisms of moduli spaces (see §10)

$$\overline{Q}_1/H_1 \simeq \overline{Q}_2/H_2,$$

$$\overline{Q}_2/H_1 \simeq \overline{Q}_3/H_2.$$

By matching on both sides of these isomorphisms the divisors corresponding to the images of $x_s = x_t$, $\{s, t\} \notin \mathscr{T}$, and also the ramifications induced by certain H_i-invariant 5-tuples, $i = 1, 2$, one can show, for example (see §§10–12, where more general results are also proved), that Γ_μ, $\Gamma_{\mu'}$, $\Gamma_{\mu''}$ are commensurable for the ball 5-tuples of the form, with $\alpha^{-1} \in \{5, 6, 7, 8, 9, 10, 12, 18\}$,

$$\mu = \left(\frac{1}{2}-\alpha, \frac{1}{2}-\alpha, \frac{1}{2}-\alpha, \frac{1}{2}-\alpha, 4\alpha\right),$$

$$\mu' = \left(\frac{1}{2}+\alpha, \frac{1}{2}+\alpha, \frac{1}{2}-2\alpha, \frac{1}{2}-2\alpha, 2\alpha\right),$$

$$\mu'' = \left(\frac{1}{2}+2\alpha, \frac{1}{2}-\alpha, \frac{1}{2}-\alpha, \frac{1}{2}-\alpha, \alpha\right).$$

For the above values of α the 5-tuples μ and μ'' satisfy $\Sigma\,$INT. The 5-tuple μ' is non-$\Sigma\,$INT for $\alpha = 1/5, 1/7, 1/9$. We shall meet these 5-tuples (up to permutation) again below in the discussion of line arrangements.

Deligne and Mostow are particularly interested in determining which lattices in $\mathrm{PU}(1, n)$ are non-arithmetic. Non-arithmetic lattices in $\mathrm{PU}(1, n)$ have been found only for $n \leq 2$, except for one commensurability class in $\mathrm{PU}(1, 3)$. This class has representative Γ_μ for μ the following ball 6-tuple satisfying $\Sigma\,$INT [M3, p. 585]

$$\mu = (3/12, 3/12, 3/12, 3/12, 5/12, 7/12).$$

They are all commensurable to lattices Γ in $\mathrm{PU}(1, n)$ which act on the complex n-ball, the quotient being an orbifold given by assigning suitable weights to the blow-up of a configuration of hypersurfaces on an algebraic variety (as in [BHH]). We shall return to this point in our discussion of line arrangements below. In [CoWo3] it was shown that it is possible to construct an embedding of the discrete not necessarily arithmetic monodromy groups Γ_μ where $n = 2$ into modular groups $\widetilde{\Gamma}$ acting on a power B_2^m of the 2-ball together with an analytic embedding (modular embedding) of B_2 into B_2^m compatible with the group embedding. The quotient $X = B_2^m/\widetilde{\Gamma}$ was shown to be a Shimura variety parametrising abelian varieties whose endomorphism algebras contain a subfield of a cyclotomic field (that is, have "generalised" complex multiplication) and $\widetilde{\Gamma}$ is in this sense a modular and hence arithmetic group. By passage to the quotient, there is a $\mathbb{Q}$-rational morphism of $\overline{Q}_\mu$ to X. The non-parabolic isolated fixed points of Γ_μ are mapped by the modular embedding to complex multiplication points in B_2^m, and this fact was used in [CoWo3] to deduce some transcendence results. The analogous construction can in principle be carried out also for $n > 2$, and the case $n = 1$ was done in detail in [CoWo1] and has applications to the theory of Grothendieck dessins [CoItzWo].

In §§15, 16, and 17 several orbifold constructions (taken from [BHH] and [Li]) are considered for surfaces and are shown to be quotients of the complex ball by lattices in PU $(1, 2)$ commensurable to a Γ_μ. We therefore report now on some results of [BHH], which include those of Höfer's dissertation, and their relevance to those sections of the book under review.

An arrangement of k lines $L_1, \dots, L_k$ in the complex projective plane has ordinary and singular intersection points. Ordinary means that exactly two lines of the arrangement pass through the point. For each line L_i we can consider the number σ_i of singular intersection points lying on it or the number τ_i of all intersection points on it. We define an endomorphism R of $\mathbb{R}^k$ [BHH, p. 182] by $R_{ij} = 3\sigma_i - 4$ if $i = j$ and $R_{ij} = 2$ if $i \neq j$ and $L_i \cap L_j$ is ordinary. Otherwise $R_{ij} = -1$. An arrangement is weighted by attaching to each line L_i a real number α_i. The weights are called admissible if the vector $(1 - \alpha_1, \dots, 1 - \alpha_k)$ is in the kernel of R. The dimension of the kernel is an important invariant of the arrangement; it vanishes if $k \neq 3$ and the arrangement has no singular points. For the complete quadrilateral, see Figure 1; the kernel has dimension 4 consisting (using the indicated notation) of all 6-tuples $\mu_i + \mu_j$ where $i \neq j$ and $1 \le i \le 4$, $1 \le j \le 4$. The singular points P_j of an arrangement are weighted by real numbers β_j satisfying

$$2\beta_j + \Sigma'\alpha_i = r - 2$$

where the sum is over the lines passing through P_j and r is the number of these lines. For the complete quadrilateral with four singular points these weights β_j are $1 - (\mu_j + \mu_5)$ where $j = 1, \dots, 4$ and $\mu_1 + \mu_2 + \cdots + \mu_5 = 2$. The kernel of R contains the subspace $(x, x, \dots, x)$ if and only if $3\tau_i = k + 3$ for each line L_i. A weighted line arrangement satisfies the condition INT if

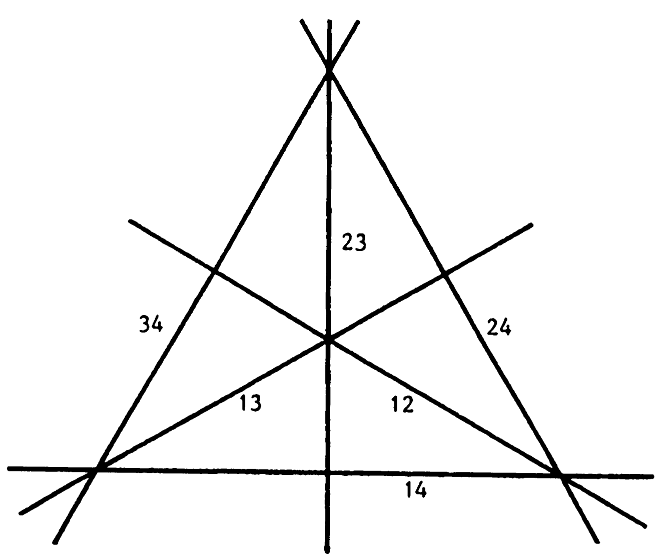

FIG. 1

all weights α_i, β_j are reciprocal integers ($1/\infty = 0$ is admitted). If we blow up all singular points of the arrangement, we get new lines E_j. We denote the lines in the blowup over L_i again by L_i and assume that INT is satisfied with all weights α_i, β_j positive. Then we can consider smooth ramified covers of the blown-up plane with ramification orders $n_i = 1/\alpha_i$ and $m_j = 1/\beta_j$ respectively. If N denotes the degree of the cover, the Chern numbers c_1^2/N and c_2/N can be calculated; they depend only on the weighted arrangement and are well defined even if such covers do not exist. If a cover is of general type, then according to Yoichi Miyaoka [Mi] and Shin-Tung Yau [Ya] we have $c_1^2 \leq 3c_2$. This inequality can be used to prove combinatorial results about line arrangements. According to Yau [Ya] a surface of general type satisfying $c_1^2 = 3c_2$ is a ball quotient, so its fundamental group is a lattice in $\mathrm{PU}(1, 2)$. Therefore, we are interested in weighted arrangements satisfying $c_1^2/N = 3c_2/N$. This formula holds if the weights are admissible. If suitable finite covers exist, for such an arrangement satisfying INT, and if some general type condition holds, then a lattice in $\mathrm{PU}(1, 2)$ is well defined up to commensurability. The general type condition is satisfied for the complete quadrilateral if $0 < \mu_i < 1$ (for $i = 1, \dots, 5$). A smooth curve D in a smooth compact ball quotient comes from a 1-dimensional subball if and only if the proportionality $2D \cdot D = e(D)$ holds where $D \cdot D$ is the self-intersection and $e(D)$ the Euler number. The formula for defining the weights β_j is equivalent to the proportionality for the curves over E_j. The admissibility of the weights α_i gives the proportionality for all curves over the lines of the given arrangement. There are 286 possibilities satisfying

$$2/m + \sum_{i=1}^{r} 1/n_i = r - 2$$

in positive integers. Still requiring admissibility and condition INT, we now do not assume anymore that all weights are positive, but in the blown-up plane the lines with non-positive weights should be disjoint. For a negative m_j or n_j we take $|m_j|$ or $|n_j|$ as the ramification index and obtain over E_j or L_j exceptional curves which can be blown down. If there are no zero weights, then $c_1^2 = 3c_2$ holds after blowing down. For weight 0 we take an arbitrary ramification index and obtain over E_j or L_j elliptic curves with negative self-intersection number. Then $3c_2 - c_1^2$ equals the sum of all these self-intersection numbers (multiplied by -1). After removing the elliptic curves we get a non-compact ball quotient if some general type condition holds. In all cases we have to use results in [ChY] and [KNS] (compare [BHH, p. 266–268]) to ensure that suitable finite covers exist which are quotients of the ball by a group of automorphisms operating freely. The general type condition can be formulated entirely in terms of the weighted arrangement. We speak of hyperbolic weights and know now that they define lattices in $\mathrm{PU}(1, 2)$. For admissible weights satisfying INT, we may get, in the non-hyperbolic case, the projective plane, or $\mathbb{C}^2$, or the product of $\mathbb{C}$ and a projective line instead of the ball. For example, the trivial weights $\alpha_i = 1$, $\beta_j = -1$ give the projective plane. The projective plane cases are 2-dimensional analogues of the 1-dimensional cases listed by H.A. Schwarz [CoWo2] (see also [Sas]).

The mirrors of a finite complex reflection group contained in $\mathrm{GL}(3, \mathbb{C})$ define a line arrangement in the complex projective plane. In this way the Ceva arrangement Ceva (q) of $3q$ lines can be obtained. In homogeneous coordinates it can be written as

$$(x_1^q - x_2^q)(x_2^q - x_3^q)(x_3^q - x_1^q) = 0.$$

The extended Ceva arrangement $\overline{\text{Ceva}}(q)$ is

$$x_1x_2x_3(x_1^q - x_2^q)(x_2^q - x_3^q)(x_3^q - x_1^q) = 0.$$

These arrangements are named after Giovanni Ceva (1647–1734) because of his theorem which gives the necessary and sufficient condition for three lines, each through one vertex of a triangle, to meet in one point. Both Ceva(2) and $\overline{\text{Ceva}}(1)$ are complete quadrilaterals. This remark has a nice application [BHH, p. 209]. We weigh each of the six lines of Ceva(2) by $2\alpha \in \mathbb{R}$ corresponding to the expression

$$x_1x_2x_3\left(x_1^2 - x_2^2\right)^{2\alpha}\left(x_2^2 - x_3^2\right)^{2\alpha}\left(x_3^2 - x_1^2\right)^{2\alpha}.$$

Introducing the three additional lines with weight 1 makes no difference. Under the map $y_i = x_i^2$ of the x-plane to the y-plane the arrangement descends to

$$y_1^{1/2}y_2^{1/2}y_3^{1/2}(y_1 - y_2)^{2\alpha}(y_2 - y_3)^{2\alpha}(y_3 - y_1)^{2\alpha}.$$

The corresponding quintuples μ for the two expressions (weighted quadrilaterals) are

$$\left(\frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ 4\alpha\right)$$

and

$$\left(\frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ \alpha,\ \frac{1}{2} + 2\alpha\right).$$

Our argument shows that we get commensurable monodromy groups if both quintuples satisfy INT. This (in the hyperbolic case) is true if and only if $\alpha = 1/p$ with $p = 6, 8, 10, 12, 18$. For $p = 5, 7, 9$ condition ΣINT holds. The commensurability is true also in this case (Sauter, see p. 82 of the book under review). In fact, as we said in our above discussion of §§10–12, Deligne and Mostow show that for these eight values of α, the quintuples

$$\left(\frac{1}{2} + \alpha,\ \frac{1}{2} + \alpha,\ \frac{1}{2} - 2\alpha,\ \frac{1}{2} - 2\alpha,\ 2\alpha\right)$$

give lattices commensurable to those above, obtaining for $\alpha = 1/5, 1/7$ two arithmetic non-ΣINT cases and for $\alpha = 1/9$ a non-arithmetic non-ΣINT case [M3, 5.5, p. 582]. In this connection let us mention the geometric interpretation in the book of the ΣINT cases with exactly three equal μ's, and suppose $\mu_1 = \mu_2 = \mu_3$. They correspond to

$$(*) \qquad (x_1x_2x_3)^\beta[(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)]^{2\alpha}.$$

The quintuple is

$$\left(\frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ \frac{1}{2} - \alpha,\ \frac{1}{2} + \alpha - \beta,\ \beta + 2\alpha\right)$$

which satisfies ΣINT if and only if α and β and also $\frac{1}{2} - \alpha - \beta$ and $\frac{1}{2} - 3\alpha$ are reciprocal integers. The condition INT is true if in addition 2α

is a reciprocal integer. Then we have orbifold covers of the complete quadrilateral in the projective plane. If Σ INT holds, we study the covering of the plane over the weighted projective plane $\mathbb{P}(1, 2, 3)$ (with Galois group $\Sigma_3 =$ symmetric group) given by $(x_1, x_2, x_3) \to (\sigma_1, \sigma_2, \sigma_3)$ where the σ_i are the elementary symmetric functions. Then $(*)$ becomes

$$(**) \qquad \sigma_3^{\beta}\, \Delta^{\alpha}$$

where Δ is the discriminant, a polynomial of weight 6 in $\sigma_1, \sigma_2, \sigma_3$. The equation $\Delta = 0$ gives in the affine plane defined by $\sigma_1 = 1$, $\sigma_2 = x$, $\sigma_3 = y$ a cuspidal cubic C which has $y = 0$ as non-cuspidal tangent L (see p. 148 and Figure 2), and $(**)$ makes geometric sense in the Σ INT-case where α is a reciprocal integer but where 2α is not necessarily a reciprocal integer (ramification index $1/\beta$ over L and $1/\alpha$ over C). The Σ INT-case $\alpha = 1/2$, $\beta = 1$ leads back to the projective plane with the complete quadrilateral. Also $\alpha = 1/3$, $\beta = 1/2$ is not hyperbolic. The "universal covering" of $\mathbb{P}(1, 2, 3)$ with ramification indices 2 and 3 along L and C respectively is the projective plane, where C corresponds to 12 and L to 9 lines, making together the (extended) Hesse arrangement which can be defined by the Hesse reflection group H of order 1296 with twelve mirrors of order 3 and nine mirrors of order 2. Observe that the quintuple for the Σ INT-case with $\alpha = 1/3$ and $\beta = 1/2$ is $(1/6, 1/6, 1/6, 1/3, 7/6)$ and the corresponding monodromy group is H (compare [CoWo2, Theorem 1], see also [Sas]). The projective group of H is of order 216. It is the automorphism group of the Hesse pencil of elliptic curves which in special coordinates can be written as

$$\lambda\left(x_1^3 + x_2^3 + x_3^3\right) + \mu x_1 x_2 x_3 = 0.$$

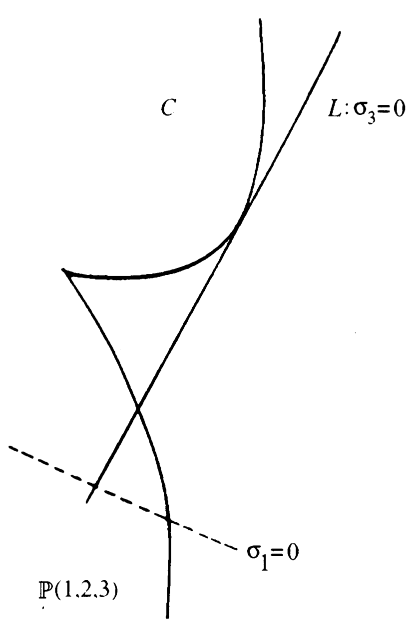

FIG. 2

The nine base points are the inflection points of all curves in the pencil. Let

$$\mathfrak{C}_{12} = x_1x_2x_3 \left[27\, x_1^3x_2^3x_3^3 - \left(x_1^3 + x_2^3 + x_3^3\right)^3\right].$$

Then $\mathfrak{C}_{12} = 0$ is the equation for the four singular elliptic curves of the pencil (4 triangles = 12 lines). The polynomial

$$\mathfrak{C}_9 = \left(x_1^3 - x_2^3\right)\left(x_2^3 - x_3^3\right)\left(x_3^3 - x_1^3\right)$$

defines the Ceva(3) arrangement which has its twelve triple-points in the vertices of the four triangles. Then $\mathfrak{C}_{12} \cdot \mathfrak{C}_9 = 0$ is the equation of the Hesse arrangement. There are fundamental invariant polynomials C_6, C_{12}, C_{18} for H. The polynomials $\mathfrak{C}_9, \mathfrak{C}_{12}$ are invariant up to factors which are roots of unity of order 2 or 3. The equations $C_{12} = 0$ and $C_{18} = 0$ give the elliptic curves in the pencil with $g_2 = 0$ or $g_3 = 0$ in the Weierstraß normal form. The map $(x_1, x_2, x_3) \mapsto (C_6, C_{12}, C_{18})$ from the projective plane to $\mathbb{P}(1, 2, 3)$ has degree 216, the equation

$$1728\, \mathfrak{C}_{12}^3 = C_{18}^2 - C_{12}^3$$

and a similar equation for $\mathfrak{C}_9^2$ show that we have come to a situation in $\mathbb{P}(1, 2, 3)$ equivalent to Figure 2. (All this invariant theory is taken from [Ma].) For the extended Hesse arrangement the kernel of the matrix R has rank 2 and the admissible weights are given by expressions which we can write as

$$\mathfrak{C}_{12}^{3\alpha}\, \mathfrak{C}_9^{2\beta}.$$

The INT-condition for the extended Hesse arrangement is that $3\alpha, 2\beta$ are reciprocal integers and

$$\frac{1}{2}(3 - 6\alpha - 6\beta)\,,\quad 1 - 6\alpha$$

are reciprocal integers $1/m_5$ and $1/m_4$ respectively where m_5 and m_4 are the ramification indices in the twelve blown-up 5-fold and in the nine blown-up 4-fold points of the Hesse arrangement. The INT-condition for the Hesse arrangement is stronger than the ΣINT-condition for (α, β), and, as Deligne and Mostow show, the lattices constructed in [BHH] from the extended Hesse arrangement are therefore commensurable to special ΣINT-cases. (See also [Y1].)

It was pointed out earlier that a line arrangement of k lines has admissible weights which are given by an arbitrary number α for all lines, if and only if k is divisible by 3 and, for each line L, the number τ of intersection points on L equals $k/3+1$. The arrangements Ceva(q), $\overline{\text{Ceva}}(q)$, the Hesse arrangement of 12 lines, and the extended Hesse arrangement of 21 lines satisfy this. In [BHH] the icosahedral arrangement, the Klein arrangement, and the Valentiner arrangement are studied. They also satisfy this condition. For them the rank of R equals 1; thus the constant weights α on all lines are the only admissible weights. The arrangements mentioned are the only ones we know that satisfy $3\tau = k + 3$ for all lines.

The icosahedral arrangement comes from the reflection group I contained in $O(3)$ of the symmetries of the icosahedron. There are only mirrors of order 2, namely, the 15 planes containing two opposite edges of the icosahedron.

The number τ equals 6; indeed on each of the 15 lines are two 2-fold, two 3-fold, and two 5-fold points of the arrangement. The order of I is 120. The fundamental invariants A, B, C of I have degrees 2, 6, 10 (see [Kl1]). There is a polynomial D of degree 15 such that $D = 0$ is the icosahedral arrangement of 15 lines and D^2 is a polynomial of weight 30 in A, B, C (see [Kl1]). A weighted icosahedral arrangement can be given by

$$D^{2\alpha}.$$

The icosahedral INT-condition is that

$$2\alpha,\ 3\alpha - \frac{1}{2},\ 5\alpha - \frac{3}{2}$$

are reciprocal integers and for $\alpha = \frac{1}{10}, \frac{1}{4}$ we get lattices in $\mathrm{PU}(1, 2)$. The methods of Deligne and Mostow admit also $\alpha = \frac{1}{3}, \frac{1}{5}$ as ramification over the curve $D^2 = 0$ in the weighted projective plane $\mathbb{P}(2, 6, 10)$. The authors announce that one gets only arithmetic lattices from the icosahedron. In a similar way we can study the simple group of order 168 acting on the projective plane. It has 21 involutions whose fixed lines are the lines of the Klein arrangement. They correspond to the 21 mirrors of a complex reflection group of order 336. The number τ equals 8. On each line there are four 3-fold and four 4-fold points of the arrangement. A polynomial K of degree 21 gives the Klein arrangement. A weighted arrangement can be expressed by $K^{2\alpha}$, and the condition INT requires that 2α, $3\alpha - \frac{1}{2}$, $4\alpha - 1$ are reciprocal integers. Indeed $\alpha = \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, 0$ give lattices. Moreover, K^2 is a polynomial in the fundamental invariants f, Δ, C of degrees 4, 6, 14 (see [W, p. 529]). The INT-condition can be relaxed to α, $3\alpha - \frac{1}{2}$, $2\alpha - \frac{1}{2}$ being reciprocal integers. This gives in addition $\alpha = \frac{1}{3}, \frac{1}{5}, \frac{1}{12}$. According to Deligne and Mostow only $\alpha = \frac{1}{3}, 0$ give arithmetic lattices.

The Valentiner arrangement has 45 lines. They come from the 45 involutions of the Valentiner group of order 360 (isomorphic to the alternating group of six letters) acting on the projective plane. Defining the α as before, $\alpha = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ give lattices. Some of the arguments concerning the icosahedral, Klein, and Valentiner arrangements are heuristic and still have to be checked.

Let us come back to the weighted arrangements $\sigma_3^{\beta}\,\Delta^{\alpha}$ in (**). For $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, and for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$ we have Euclidean cases, and suitable finite covers are abelian surfaces. For $\alpha = \frac{1}{8}$, $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{9}$, $\beta = \frac{1}{3}$ we can construct coverings of abelian surfaces which are ball quotients. Deligne and Mostow show that these are the ball quotients studied in [BHH, §1.4], up to commensurability. Also for $\alpha = 0$, $\beta = \frac{1}{3}$ we get coverings of an abelian surface which are (non-compact) ball quotients, namely, those studied in §1 of [Hi]. The quintuple is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. In [CoWo2] the complete list of the discontinuous Euclidean Appell-Lauricella monodromy groups is given (completing those found in [BHH] and in this book), that is, the discontinuous Γ_μ with μ an $(n+3)$-tuple $(\mu_1, \dots, \mu_{n+3})$ of rational numbers with $\sum_{i=1}^{n+3} \mu_i = 2$ with just one of the μ_i an integer.

In §16 Deligne and Mostow relate Livne's construction of lattices in $\mathrm{PU}(1, 2)$ to groups $\Gamma_{\mu, H}$. For an integer $n \geq 3$ and d an integer satisfying $d|n$ if n is odd and $d|\frac{n}{2}$ if n is even, certain cyclic covers $\tilde{E}_d(n)$ of the Shioda elliptic modular surface $E(n)$ of level n can be constructed. The cyclic cover

$\tilde{E}_d(n) \to E(n)$ has order d and is ramified over the n-division point sections of the elliptic fibration $E(n) \to X(n)$. When $n = 7, 8, 9, 12$ and $d = \frac{n}{n-6}$, Livne [Li] and Inoue (see Acknowledgments) showed independently that $c_1(S)^2 = 3c_2(S)$ for $S = \tilde{E}_d(n)$, so that the surface $\tilde{E}_d(n)$ is a compact ball-quotient. Let A be the automorphism group of $\tilde{E}_d(n)$. In §16 Deligne and Mostow show that for $n \geq 3$, the quotient $\tilde{E}_d(n)/A$ is the moduli space of a projective line, a marked point 0, and an unordered set $\mathscr{A}$ of three points A', A'', A''' and an additional point x, where one allows coincidences between two elements of $\mathscr{A}$, between x and 0, and between x and any one or two elements of $\mathscr{A}$. Hence as a moduli space $\tilde{E}_d(n)/A = \overline{Q}_\mu/\Sigma$ where $\mu = (\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}, \frac{1}{n}, \frac{1}{2} + \frac{2}{n})$, $\Sigma = \Sigma\{1, 2, 3\}$, the permutation group on $\{1, 2, 3\}$, for any integer $n > 6$. Now, μ satisfies ΣINT for $n > 4$ if and only if $\tau = \frac{2n}{n-6} \in \mathbb{Z}$, that is, $n \in \{5, 6, 7, 8, 9, 10, 12, 18\}$. On the other hand, the ramification divisors of $\tilde{E}_d(n)/A$ with their ramifications are: $x = 0$, ramification index $2d$; $x = A', A'', A'''$, ramification index 2; two of the elements of $\mathscr{A}$ coincide, ramification index n. Hence, for $n \in \{7, 8, 9, 12\}$ we have $\tilde{E}_d(n)/A = \overline{Q}_\mu/\Sigma = B_2/\Gamma_{\mu,\Sigma}$ as orbifolds. In the remaining cases in which μ satisfies ΣINT, Deligne and Mostow show how the quotient $\tilde{E}_d(n)/A$ may be modified to arrive at an orbifold isomorphic to $\overline{Q}_\mu/\Sigma$, and they treat also the non-hyperbolic cases $n = 3, 4$.

The moduli space mentioned above is related to $\mathbb{P}(1, 2, 3)$; see Figure 2. This is induced by the map $E(n) \to \mathbb{P}(1, 2, 3)$ given by $\wp(nz) : g_2 : g_3$ (Jacobi forms of weights 2, 4, 6 respectively). The equation

$$\wp'(nz)^2 = 4\wp(nz)^3 - g_2\wp(nz) - g_3 = 0$$

corresponds to the line L in Figure 2 and $g_2^3 - 27g_3^2 = 0$ to the cubic curve C (discriminant). The fibres of $E(n)$ go to the pencil $\alpha g_2^3 - \beta g_3^2 = 0$ of cuspidal cubic curves; the n^2 sections of $E(n)$ (poles of $\wp(nz)$) collapse to the point $1 : 0 : 0$. The weighted arrangement $\sigma^\beta \Delta^\alpha$ (see (**)) has been mentioned very often. For $\alpha = \frac{1}{n}$, $\beta = \frac{1}{2}$ we get the lattices of the previous paragraph. For $n = 3$ the map is given by $C_6 : C_{12} : C_{18}$, and $E(3) = \tilde{E}_1(3)$ is the Hesse pencil (i.e. the projective plane with the nine base points of the Hesse pencil blown up). The Shioda modular surface $E(4)$ is a K3-surface. The ramified cover $\tilde{E}_2(4)$ of degree 2 along the 16 sections (rational curves of self-intersection -2) gives one of the abelian surfaces mentioned earlier (with the 16 two-division points blown up). By a result of Ishida [I] the surface $\tilde{E}_5(5)$ (all 25 sections collapsed) is a 125-fold covering of the projective plane along the complete quadrilateral corresponding to the quintuple $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$. Therefore $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5})$ and $(\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{4}{5})$ give commensurable lattices, a fact not occurring in the book as far as we can see.

Acknowledgments

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