theory and its applications to geometry; this includes concise outlines of classical results—Hopf's and Alexandrov's theorems on constant mean curvature spheres and the Ruh-Vilms Theorem—and of Wente's construction of a constant mean curvature torus (following an approach by Abresch). The authors then discuss the equivariant setup. The cases where the quotient of the domain space by the group action is zero- or 1-dimensional are treated in detail. The former leads to the result that any compact homogeneous space is harmonically (and minimally) immersible into some Euclidean sphere. The latter case yields the construction (by studying the corresponding ODE's) of special $SO(n-1)$-invariant minimal hypersurfaces in S^n, of a countable family of embedded $SO(2) \times SO(2)$-invariant minimal 3-spheres in S^4, and of a countable family of embedded $SO(p) \times SO(q)$-invariant constant mean curvature hyperspheres in $\mathbb{R}^{p+q}$, $p, q \geq 2$. The remaining third of the book deals with the construction of harmonic maps from spheres into spheres. The basic idea to obtain such maps is by "joining" eigenmaps. The harmonicity equation reduces to a pendulum-type equation (with variable damping and gravity) for the "joining" function, whose qualitative global behaviour can be studied.

Except for this last part, which is mostly a topic for the expert reader, the book is written for a general audience with some interest in harmonic maps. Each chapter concludes with a welter of notes and comments, giving the reader a more global context as well as many hints for further studies. The pace of the book is comfortable, even when it comes to technical lemmas, which are kept balanced by qualitative statements. To avoid losing focus, the reader is provided with a detailed roadmap at the beginning of the book and of each chapter. The authors conclude with a rather comprehensive reference list. Parts of the book can certainly form the basis for a topic course, and it could be suggested reading for graduate students and researchers with an interest in harmonic maps.

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Invariant distances and metrics in complex analysis, by Marek Jarnicki and Peter Pflug. Expositions in Mathematics, vol. 9, Walter de Gruyter, Berlin, 1993, xi+408 pp., $98.95. ISBN 3-11-013251-6

The subject of invariant metrics in complex analysis started with Poincaré's investigations of Fuchsian groups, that is, discrete subgroups in the group of all holomorphic automorphisms of the unit disc $E \subset \mathbb{C}$. Poincaré made two crucial observations: first, if one defines the length of a smooth curve $\Gamma \subset E$ by

$$\int_\Gamma \frac{|dz|}{1-|z|^2}$$

and the distance of two points a, $b \in E$ as the length of the shortest curve joining a, b, a metric $p(a, b)$ is obtained on E that is invariant under holomorphic automorphisms of E. Explicit formulae are easily obtained for $p(a, b)$; for example, $p(0, b) = \frac{1}{2}\log\frac{1+b}{1-b}$. The second observation was that E with this metric p is isometric to the hyperbolic (or non-Euclidean) plane. Thus Fuchsian groups appear as discrete groups of motions of the hyperbolic plane, and their structure therefore can be investigated using Euclidean analogies.

It is curious that in his papers on the subject Poincaré only sparingly mentions non-Euclidean geometry, yet its importance as a guiding principle is most apparent in his work on Fuchsian groups. It is even more curious that although he later became interested in geometric function theory in several variables, he never returned to the subject of invariant metrics. Had he done so, he would have had at his disposal a powerful tool in the study of holomorphic and biholomorphic (= invertible and holomorphic) mappings in several complex variables.

As it happened, not until the 1920s were metrics introduced on domains in $\mathbb{C}^n$, metrics that were invariant under biholomorphic mappings. Then Stefan Bergman and Constantin Carathéodory invented two metrics that were later named after them. In fact, in both cases we are talking about a family of metrics, one for each bounded domain $G \subset \mathbb{C}^n$; biholomorphic invariance means (in the case of the Carathédory metric c_G) that if $f: G \to H$ is a biholomorphic mapping between bounded domains in $\mathbb{C}^n$, then

$$c_H(f(z), f(w)) = c_G(z, w), \qquad z, w \in G$$

(and similarly for the Bergman metric b_G). This very easily follows from the definition of the metrics, which we give here only for the Carathéodory metric:

$$c_G(z, w) = \sup p(f(z), f(w)),$$

taken over all holomorphic $f: G \to E$. Both Bergman and Carathédory invented their metrics in order to study biholomorphic equivalence of domains. Success was moderate though; early researchers found it difficult to bridge the gap between the abstract definition of the metrics with the geometric properties of domains. For example, there were very few domains for which the metrics could be explicitly computed, and it was not clear at all how the invariance property could be exploited if explicit formulae were not available.

Accordingly, the subject languished until the sixties and early seventies, when three major (and independent) advances occurred. One was the theory of the inhomogeneous Cauchy-Riemann equations on noncompact complex manifolds, due to Kohn and Hörmander. This had immediate applications to the Bergman metric, if not to biholomorphic mappings at that point. The second major event was the publication of Kobayashi's book, *Hyperbolic manifolds and holomorphic mappings*, where what is now called the Kobayashi metric was introduced. This is defined in a way dual to Carathéodory's: First one puts

$$\widehat{k}_G(\lambda, \mu) = \inf p(\lambda, \mu),$$

with λ, $\mu \in E$ such that there is a holomorphic map $f: E \to G$ that takes λ, μ to z, w. This does not always satisfy the triangle inequality, so one then takes the largest (pseudo)metric less than $\hat{k}_G$ and calls it the Kobayashi (pseudo)metric k_G. (The prefix "pseudo" refers to the possibility that two distinct points may be at distance zero, which indeed may occur if the domain in question is not bounded, or if one looks at general complex manifolds, for which the definitions of Bergman, Carathéodory, and Kobayashi automatically extend.)

Finally, it was observed essentially simultaneously by Vormoor and Henkin that the so-called contracting property of the Carathéodory metric $c_G(z, w)$ can be combined with its known asymptotic behavior (as one or both points z, w tend to the boundary ∂G) to show that biholomorphic mappings extend to homeomorphisms between the closures of (bounded, strongly pseudoconvex) domains. The contracting property in question means that if $F: G \to H$ is any holomorphic (not necessarily invertible) mapping, then

$$c_H(F(z), F(w)) \leq c_G(z, w).$$

The Kobayashi metric k_G also enjoys this property, while the Bergman metric does not.

The three major events referred to above have become a prelude to a very active and fruitful period of research. The subject developed along two lines. One was value distribution theoretic, the direction started by Kobayashi and motivated by Picard's theorem. Here one looks at complex manifolds (or complex spaces) M and N and asks whether any holomorphic map $f: M \to N$ has to be constant—as in the situation considered by Picard, when $M = \mathbb{C}$ and N is $\mathbb{C}$ minus two points. Because of the contracting property of the Kobayashi metric, this will be the case if we know that $k_M \equiv 0$ while $k_N(z, w) = 0$ only if $z = w$. Accordingly, it is of utmost importance to understand for which manifolds N is the Kobayashi metric positive off the diagonal; in which case N is called hyperbolic. The search for hyperbolic manifolds is a rich source of open problems and deep results, with connections to complex differential geometry, algebraic geometry, and number theory.

The other direction of research in the subject of invariant metrics was aimed at describing finer properties of invariant metrics, with an eye on the study of biholomorphic mappings and, more generally, of holomorphic invariants. As said above, for value distribution theory only $k_N(z, w)$ being zero or not is what matters; by finer properties we mean—e.g., in the case of the Kobayashi metric—how smoothly $k_N(z, w)$ depends on z, w, and what is its asymptotic behavior as z, w tend to (some ideal) boundary of N; what can be said about the geodesics or the curvature of k_N, etc. To obtain reasonable answers, one often has to restrict to manifolds that can be realized as domains in some complex Euclidean space; moreover, these domains are assumed to have smooth boundaries and enjoy some degree of (pseudo)convexity. For such domains there is a rich theory of invariant metrics and some tantalizing open problems.

The book under review by Jarnicki and Pflug deals entirely with this second aspect of invariant metrics. (For those interested in value distribution theory, the books by Lang, *Introduction to complex hyperbolic spaces* (Springer, 1987); and Noguchi and Ochiai, *Geometric function theory in several complex variables* (AMS, 1990), can be recommended.) The book offers a thorough study of some basic properties of a handful of invariant metrics that have been introduced in addition to the metrics mentioned above over the past twenty years or so. Given this scope, it does a very good job of presenting the state of the art. The style is clear and concise; many examples illustrate the general theory; and, overall, the book makes pleasant reading. It can be profitably used by graduate students who, after an introductory course in several complex variables, will undoubtedly welcome the numerous challenging open problems sprinkled all over the text.

(In parenthesis, the reviewer expresses his dislike for two pieces of terminology that are rather widespread in the subject but incorrect. Unfortunately, the authors chose to adopt them in their monograph. One is the notion of "contractible" metrics. The suffix "-ible" refers to some possibility, such as in contractible topological spaces, that can be contracted to a point, if need be. However, the metrics in question cannot be contracted in any sense of the word; rather they will mandatorily contract, if a holomorphic mapping is applied. Perhaps contracting metrics is a better term; other names are also in use, such as Schwarz systems.

The other term objected to is "complex ellipsoids". These are not ellipsoids at all, not even "complex analogs" of ellipsoids. They have become a popular testing ground for explicit computations in complex analysis, so perhaps they do deserve a name of their own. Some simply call them egg domains, which is more appropriate than the slightly misleading term used in the book.)

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Floquet theory for partial differential equations, by Peter Kuchment. Operator Theory Advances and Applications, vol. 60, Birkhäuser Verlag, Basel and Boston, 1993, xiv+350 pp., $108.50. ISBN 0-8176-2901-7

How does periodicity in a differential equation show up in the solutions? While a solution is rarely itself periodic, it surely bears marks of the symmetries in the equation it solves. This question was answered for linear ordinary differential equations in 1883 by G. Floquet [Fl]: As the variable increases through the period ω, there are basic solutions which he described as "periodic of the second kind", i.e., $u(t+\omega) = \varepsilon u(t)$ for some multiplier ε, which is usually not 1. In contemporary terminology, Floquet's theorem concerns an ω-periodic $n \times n$ matrix-valued function, say continuous, $P(t)$ and the equation

$$y' = Py. \tag{1}$$

This embraces higher-order differential equations by a familiar transformation. The n-dimensional general solution is determined by a fundamental matrix $Y(t)$, and the theorem states that Y can be written in the form

$$Y(t) = Z(t)\exp(Rt), \tag{2}$$

where Z is periodic with period ω and where R is a constant matrix. (For instance, see [Ha].) A natural spectral problem for the ordinary differential equation is the determination of Z and R, especially the eigenvalues of R.

Although periodic structures are common in nature and Floquet theory has always had many applications in physics, Floquet himself apparently came upon