who, after an introductory course in several complex variables, will undoubtedly welcome the numerous challenging open problems sprinkled all over the text.

(In parenthesis, the reviewer expresses his dislike for two pieces of terminology that are rather widespread in the subject but incorrect. Unfortunately, the authors chose to adopt them in their monograph. One is the notion of "contractible" metrics. The suffix "-ible" refers to some possibility, such as in contractible topological spaces, that can be contracted to a point, if need be. However, the metrics in question cannot be contracted in any sense of the word; rather they will mandatorily contract, if a holomorphic mapping is applied. Perhaps contracting metrics is a better term; other names are also in use, such as Schwarz systems.

The other term objected to is "complex ellipsoids". These are not ellipsoids at all, not even "complex analogs" of ellipsoids. They have become a popular testing ground for explicit computations in complex analysis, so perhaps they do deserve a name of their own. Some simply call them egg domains, which is more appropriate than the slightly misleading term used in the book.)

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Floquet theory for partial differential equations, by Peter Kuchment. Operator Theory Advances and Applications, vol. 60, Birkhäuser Verlag, Basel and Boston, 1993, xiv+350 pp., $108.50. ISBN 0-8176-2901-7

How does periodicity in a differential equation show up in the solutions? While a solution is rarely itself periodic, it surely bears marks of the symmetries in the equation it solves. This question was answered for linear ordinary differential equations in 1883 by G. Floquet [Fl]: As the variable increases through the period ω, there are basic solutions which he described as "periodic of the second kind", i.e., $u(t+\omega) = \varepsilon u(t)$ for some multiplier ε, which is usually not 1. In contemporary terminology, Floquet's theorem concerns an ω-periodic $n \times n$ matrix-valued function, say continuous, $P(t)$ and the equation

$$y' = Py. \tag{1}$$

This embraces higher-order differential equations by a familiar transformation. The n-dimensional general solution is determined by a fundamental matrix $Y(t)$, and the theorem states that Y can be written in the form

$$Y(t) = Z(t)\exp(Rt), \tag{2}$$

where Z is periodic with period ω and where R is a constant matrix. (For instance, see [Ha].) A natural spectral problem for the ordinary differential equation is the determination of Z and R, especially the eigenvalues of R.

Although periodic structures are common in nature and Floquet theory has always had many applications in physics, Floquet himself apparently came upon it through pure curiosity about the effect of symmetries on the structure of solutions. The inspiration for his investigations was the theory of linear ordinary differential equations in the complex plane, especially the work of I. L. Fuchs and M. Hamburger on integrals of solutions around singularities. He was unaware that a technique for detailed calculations of some of the multipliers he defined had been discovered several years earlier by G. W. Hill [Hi], who was solving second-order equations of the specific form

$$w'' = \Theta w$$

where Θ is a periodic function. Hill was not proving theorems but was rather describing the theory of the precession of the moon's orbit and calculating it systematically from the Fourier-series development of Θ.

There are two conceptually distinct ways that Floquet theory can arise for the partial differential equations of physics. The spatial structure might be periodic, as in a crystal, or some force might be periodic in time, as in Hill's problem. Spatial periodicity should have been the more difficult advance, since the period becomes a higher-dimensional quantity, a vector in the reciprocal lattice. Floquet's approach to the subject was essentially a spectral analysis of the Floquet operator, also known as the monodromy operator or period operator, corresponding to knowing $Y(\omega)$ in the notation above. There is not close analogue of the Floquet operator for higher-dimensional periodicity.

Nonetheless, Floquet theory arose earlier in partial differential equations in spatially periodic situations. The landmark paper, on quantum mechanics in crystals, was published in 1929 by my teacher F. Bloch [Bl]. As a brash young physicist in the frontier days of quantum mechanics, Bloch had little use for the mathematical literature and did not even refer to Floquet. If there was a problem to solve, he preferred to invent anew any mathematics needed. In this article he used group theory, rather popular at the time, to conclude that the eigenfunctions of the time-independent, spatially periodic Schrödinger equation

$$(-\nabla^2 + q(\mathbf{x}) - \lambda)\psi(\mathbf{x}) = 0 \tag{3}$$

were of the form $\exp(i\langle \mathbf{k} \cdot \mathbf{x}\rangle)u(\mathbf{x})$, where u is periodic and $\mathbf{k}$ is a reciprocal vector, known as the *quasimomentum*. There is a significant theorem here, though it was not originally written in a careful mathematical style. As Kuchment presents Bloch's theorem in his book, it states that if there is a nonzero bounded solution, then there exists a solution of the form given by Bloch and there is correspondingly a point of the L^2 spectrum of the Schrödinger operator

$$L := -\nabla^2 + q \tag{4}$$

at the level λ. Indeed, by a result of I. E. Shnol′, we need only assume that the solution is subexponential at infinity. The important issue of completeness of the Bloch solutions was neglected until 1950, when I. M. Gel′fand [Ge] introduced the Gel′fand transform for this purpose. Later work on completeness was done by V. B. Lidskii and by F. Odeh and J. B. Keller [OK].

The spectral theory of (3) is most sensibly approached using both parameters $\mathbf{k}$ and λ together. The *Bloch variety* can be defined as the set in C^{n+1} of parameters for which there is a Bloch solution, and its structure turns out to be that of the zero set of a certain entire function of several complex variables. This set, or the variant called here the Floquet variety, should be a key element in the inverse spectral problem for (3). The basic analytic structure of these sets is described by Kuchment, but, as he points out, many interesting structural issues, such as irreducibility, are still open.

An alternative to studying the Bloch variety is to look at the band functions, i.e., individual eigenvalues of (3) as they depend on the quasimomentum, $\lambda_n(\mathbf{k})$, and this is the point of view that physicists have found more congenial. For each fixed $\mathbf{k}$, the reduced Bloch Hamiltonian has discrete spectrum. The full operator L is a direct integral over $\mathbf{k}$ of the reduced Hamiltonians, and consequently the spectrum of L is the union of the spectra of the reduced Hamiltonians. The fundamental theorems about the band functions and reduced Hamiltonians were proved by L. E. Thomas [Th] and by J. E. Avron, A. Grossmann, and R. Rodriguez [AGR1–2] in the 1970s. (The articles by the latter are among the few significant items missing in Kuchment's bibliography.) In particular, Avron, Grossmann, and Rodriguez showed the compactness of the reduced resolvents and obtained the asymptotic distribution of the band functions of (4) (and various generalizations), which is independent of q under wide circumstances.

In one dimension, i.e., Hill's equation, if there are only finitely many gaps in the spectrum, then the finite-dimensional manifold of potentials q with the spectrum can be computed. In particular, if the spectrum is $[0, \infty)$, then, uniquely, $q(x) = 0$ a.e. While some pieces of the one-dimensional theory survive into more dimensions, the situation is less understood and certainly more complicated, since there are lots of many-dimensional potentials with no spectral gaps. Indeed, the generic situation for the Schrödinger equation in many dimensions is that there can be only a finite number of gaps in the spectrum. If the variables can be separated, then this is easy to see, because the spectrum of the partial differential equation is the sum (as a set) of spectra of the constituent ordinary differential equations and in more than one dimension the band functions are asymptotically closely spaced in comparison with the widths of the bands as $\mathbf{k}$ varies over a fundamental cell. Early solid-state physicists such as H. Bethe and A. Sommerfeld believed that the same state of affairs must persist for nonseparable equations, but this fact waited decades for a satisfactory rigorous proof by M. M. Skriganov [Sk], who related it to delicate number-theoretic estimates about the ranges of certain quadratic forms on integers.

Time-periodic parabolic partial differential equations have been studied in various ways by various people, including A. I. Miloslavskii [Mi] and Kuchment [Ku], who devotes a chapter in his book to these and their generalization as hypoelliptic evolution equations. His interest is again mainly in the construction of solutions of Floquet type and their completeness for Cauchy problems. Floquet theory had earlier been developed for equations of the form (1) with P replaced by a bounded operator on an abstract space in [MS] and [DK], but this restriction excludes partial differential equations. In the late 1970s, progress was made on these questions by replacing the study of the Floquet operator for (1) by the study of operators like

$$K = \frac{d}{dt} - P$$

on augmented spaces such as $L^p((0, \omega); \mathscr{H})$, consisting of maps from the period interval to a Hilbert space. This was also done by Yajima [Ya] and Howland [Ho1], who used techniques of scattering theory to study Schrödinger equations periodic in time

$$i\frac{\partial \psi}{\partial t} = H(t)\psi ,$$

where for each t, H is self-adjoint on $\mathscr{H}$. It is possible to construct scattering operators, carry out perturbation theory, and much else which is good and useful.

The book by Kuchment offers many interesting perspectives on periodic problems and the structure of the operator L of (4). Several theorems can be proved for more general types of equations, especially on the completeness of Bloch or Floquet solutions and expansion in them. Kuchment's chapters on the solutions and spectral properties of spatially periodic elliptic or hypoelliptic equations constitute the heart of his book, and this subject is covered in more depth here than in any other available book. Classically oriented readers may be frightened in the introductory chapter by discussion of "holomorphic subbundles of holomorphic Banach bundles over Stein manifolds" but will be reassured later when the discussion of the structure and completeness of Bloch and Floquet solutions gets more specific. The book is well organized and generally well written, except for an occasional missing definite or indefinite article.

The greatest omission of Kuchment's book is that he does not discuss the nature of the spectrum, only its location. Techniques and concepts from scattering theory have in fact been successfully used to investigate the structure of the spectrum in both spatially and temporally periodic problems. It is known, for instance, that if the Fourier series coefficients of q are in certain l_p classes, then the spectrum of L in (4) is purely absolutely continuous (see Chapter XIII of [RS]). In striking contrast, Howland [Ho2–3] has shown that the Floquet operators for large classes of quite ordinary time-periodic Schrödinger equations have no absolutely continuous spectrum at all. It would not have been difficult to include some of these topics and gain a broader perspective. Nonetheless, the book is a very useful resource for those interested in partial differential equations with symmetries and contains much that is not available elsewhere.

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Duality and perturbation methods in critical point theory, by Nassif Ghoussoub. Cambridge University Press, Cambridge, 1993, xviii+258 pp., $49.95. ISBN 0-521-44025-4

There are many problems in mathematics and physics, for instance in geometry and mechanics, whose solutions are extrema of some kind of energy. Geodesics on Riemannian manifolds, minimal surfaces, conformal metrics with prescribed curvature properties, harmonic maps, as well as solutions to Hamiltonian systems or semilinear elliptic partial differential equations are critical points of a suitable energy (or action) functional. The least action principle of Euler and Maupertuis even expresses the belief that every law of nature can be formulated as an extremum principle.

The classical calculus of variations whose origins can be traced back to Fermat, Newton, and the Bernoulli brothers deals with the problem of studying properties of an extremum—once the appropriate "energy" has been found. The Euler-Lagrange equations which are necessarily satisfied by any extremum have been investigated by many mathematicians ever since their discovery in